Tutorial for

Introduction to Computational Intelligence in Winter 2015/16

Günter Rudolph, Vanessa Volz

Lecture website: `https://tinyurl.com/CI-WS2015-16`

Sheet 2, Block I

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Due date: 25 November 2015, 2pm
Discussion: 26/27 November 2015

Exercise 2.1: Radial Basis Function Nets (6 Points)

Implement an RBF network with Gaussian basis function in R. Alternatively, find, download, understand and describe a public domain version. Use that implementation to model the data set given in `data.csv` (first two columns input, third column class).

- Elaborate on your choice for the number of neurons q, the radii σ and the center c_k.
- Visualise (in one or more plots):
  - $\Phi(x; c)$ for all neuron center c_k and $\forall (x, y) \in [-2, 2] \times [-2, 2]$
  - classification results and errors
- Using your visualisations, analyse your classification results and try to explain any shortcomings.

Exercise 2.2: Weights for RBF (8 Points)

The optimal weights $\mathbf{w}$ for an RBF net can be determined from the solution of the matrix equation $P\mathbf{w} = \mathbf{y}$ via the pseudo inverse of P.

a) Show formally that the optimal weights can be determined via minimizing
$\|P\mathbf{w} - \mathbf{y}\|^2 = (P\mathbf{w} - \mathbf{y})'(P\mathbf{w} - \mathbf{y}) \to \text{min!}$

   Use differential calculus.

b) If the training examples lead to an ill-conditioned matrix P the numerical process can be made more stable if we minimize the objective function
$\|P\mathbf{w} - \mathbf{y}\|^2 + \mathbf{w}'D\mathbf{w} \to \text{min!}$,
where $D = \text{diag}(d_1, \ldots, d_q)$ is a diagonal matrix with positive diagonal entries $d_i > 0$.

   Derive the expression for the optimal weights via differential calculus.

Solution (shortened version!)

a)

$$\begin{aligned}
||Pw - y||^2 &= (Pw - y)^T(Pw - y) \\
&= w^T P^T pw - y^T Pw - P^T w^T y + y^T y \\
&= w^T P^T pw - 2w^T P^T y + y^T y
\end{aligned}$$

Figure 1: Learning patterns (Exercise 2.3)

Figure 2: Test patterns (Exercise 2.3)

$$
\begin{aligned}
\frac{\delta}{\delta w}||Pw - y||^2 = 2P^T Pw - 2P^T y \stackrel{!}{=} 0 \\
\Rightarrow P^T Pw = P^T y \\
\Rightarrow w = (P^T P)^{-1} P^T y \\
\Rightarrow w = P^+ y
\end{aligned}
$$

b)

$$||Pw - y||^2 + w^T Dw = w^T P^T pw - 2w^T P^T y + y^T y + w^T Dw$$

$$
\begin{aligned}
\frac{\delta}{\delta w}|Pw - y||^2 + w^T Dw = w^T P^T pw - 2w^T P^T y + y^T y + w^T Dw \stackrel{!}{=} 0 \\
2P^T Pw + 2Dw - 2P^T y = 0 \\
P^T Pw + Dw - P^T y = 0 \\
P^T y = P^T Pw + Dw \\
P^T y = (P^T P + D)w \\
w = (P^T P + D)^{-1} P^T y
\end{aligned}
$$

Exercise 2.3: Hopfield Nets for Error Correction (6 Points)

Assume a Hopfield net with 9 neurons, aranged in a 3×3 grid, which results in a weight matrix with 81 entries.

Using associative memory, we want to store two different patterns (visualised in figure 1) in the network. The goal is the following: if a slighlty modified version of either pattern is fed to the network, after a few iterations, the network displays the corresponding stored pattern, thus *correcting* the input.

Storing the patterns can be achieved by using the *Hebb rule* (equation 1) to initialise the network weights. This way, the stored patterns are attractors (stable states) in the resulting energy landscape and can thus be retrieved.

Let m be the number of patterns to be stored in the network and n be the number of neurons in the network. x_i^μ, with $\mu \in [1, m], k \in [1, n]$, is the state of neuron i (either 1 or -1) in pattern μ.

According to Hebb's rule, the initial weights should then be:

$$W_{ij} = \frac{1}{n} \sum_{\mu=1}^{m} x_i^\mu x_j^\mu \tag{1}$$

(For more background information, look up Hopfield Nets and Hebbian learning)

- Implement the Hopfield Net described above and store the patterns in figure 1 by initialising the weights according to equation 1. Print the weight matrix.
- Now use the patterns depicted in figure 2 as input. In how many iterations does the network reach a stable state? What pattern does it retrieve? Explain the network's behaviour.
- What is the effect on the energy function if for each pattern μ a weight $\epsilon_\mu \in \mathbb{R}$ is factored into the weight initialisation in equation 1, such that:

$$W_{ij} = \frac{1}{n} \sum_{\mu=1}^{m} \epsilon_\mu x_i^\mu x_j^\mu \tag{2}$$