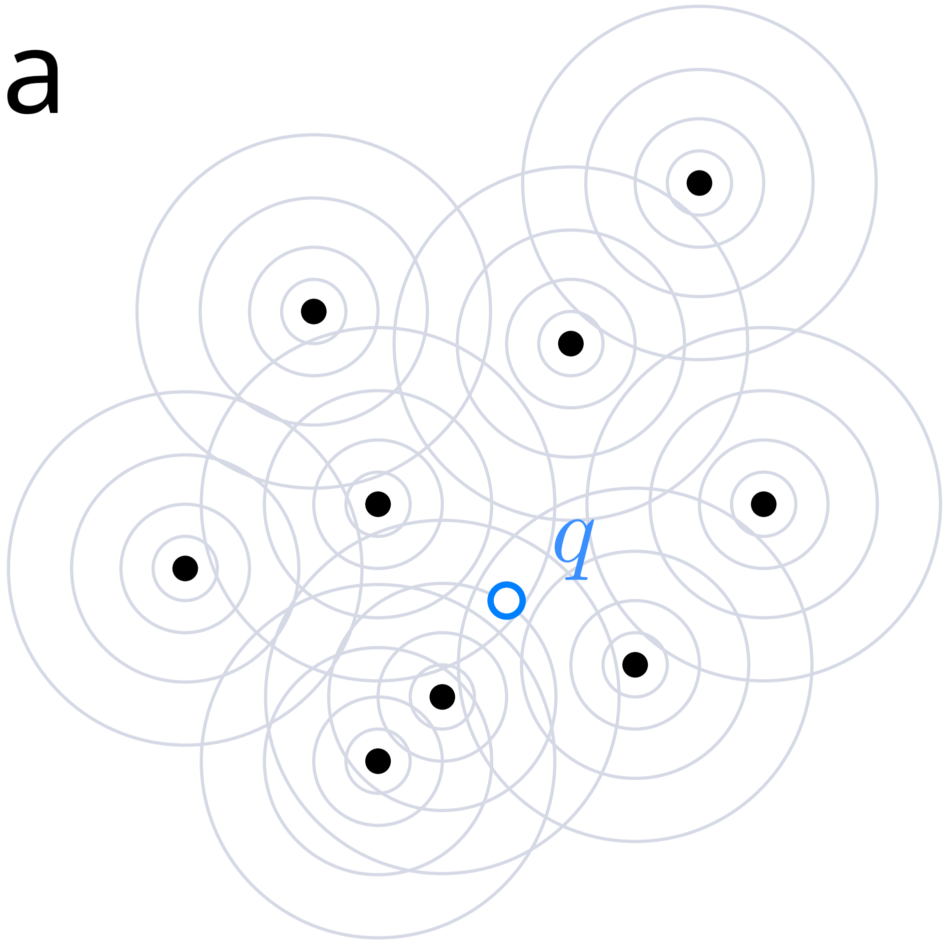
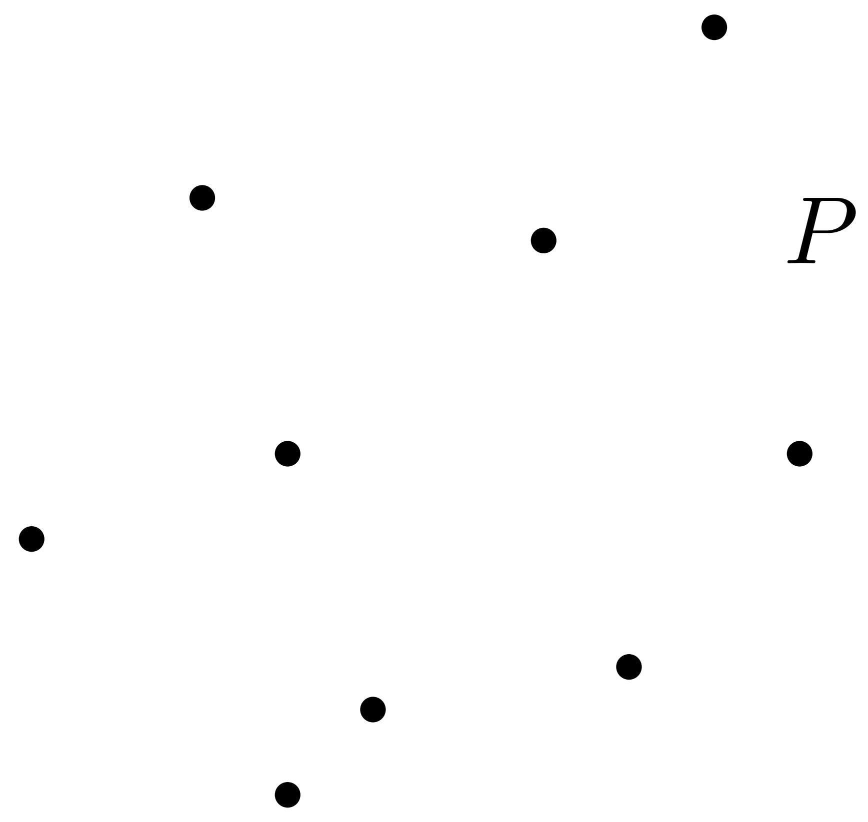
Approximate Nearest Neighbors via Point Location Among Balls



ANN: What happened so far

$(1+\varepsilon)$**- Approximate Nearest Neighbors (ANN)**



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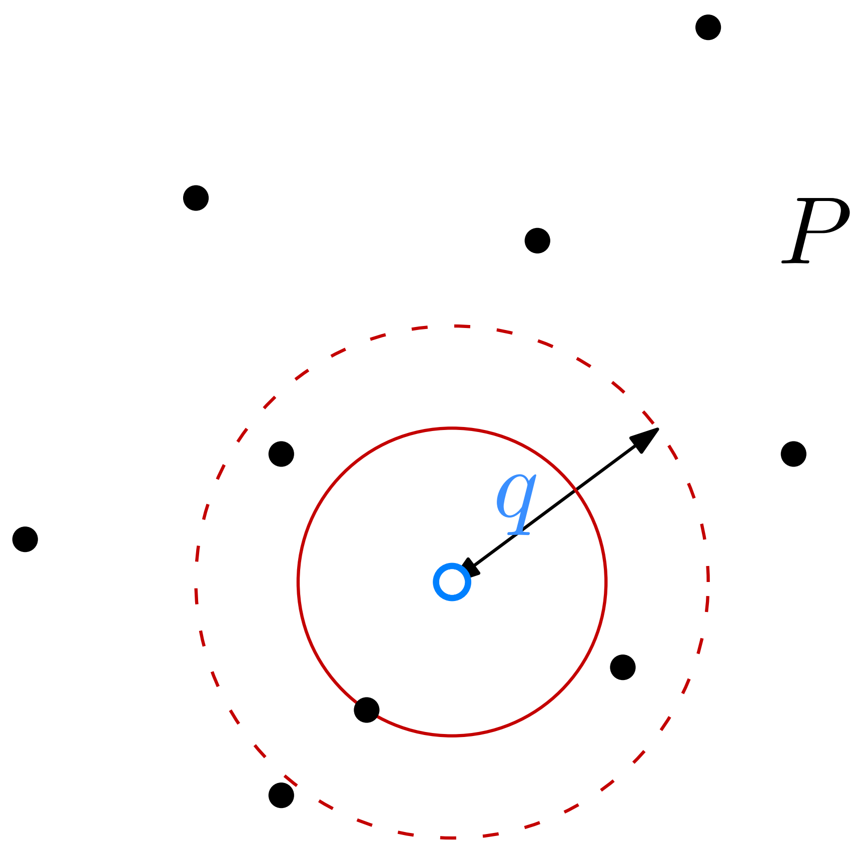

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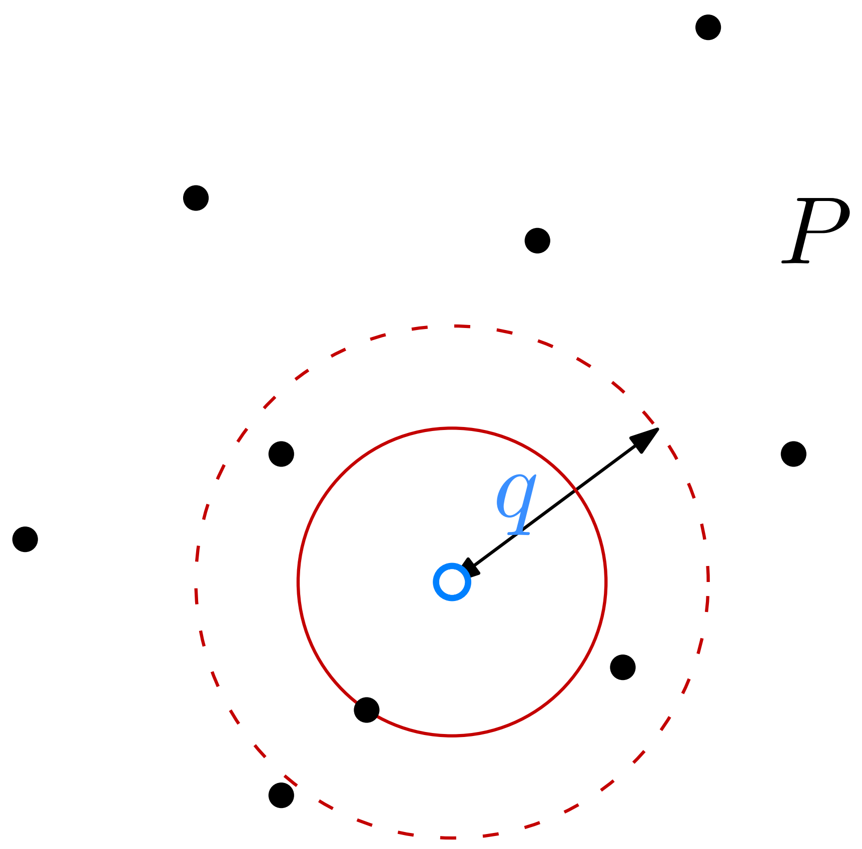

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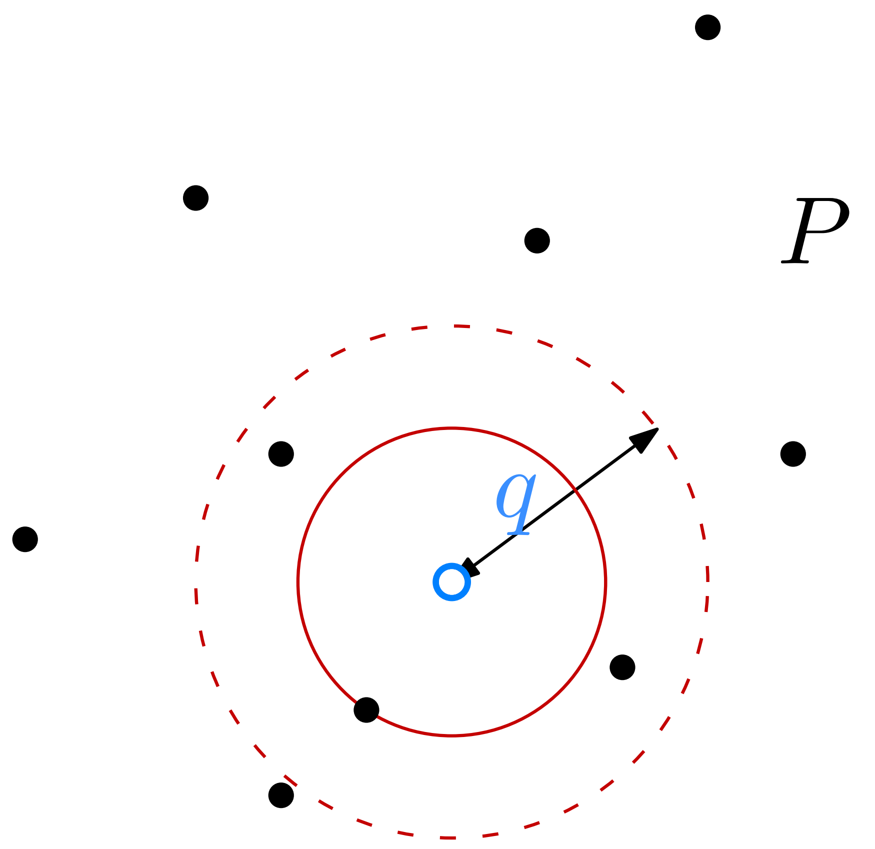

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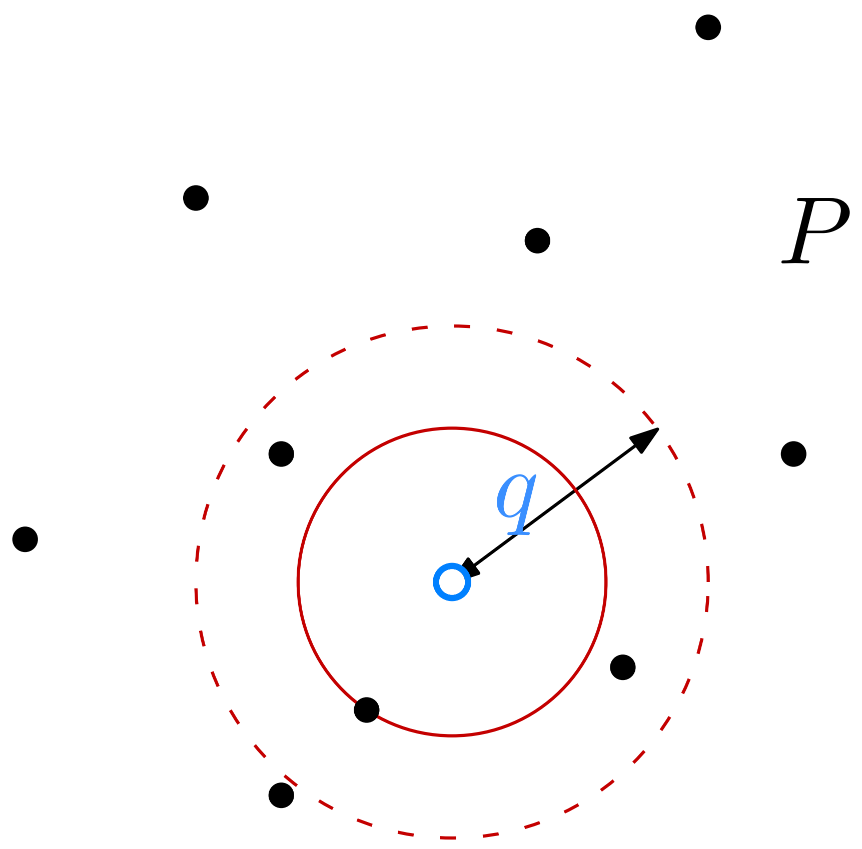

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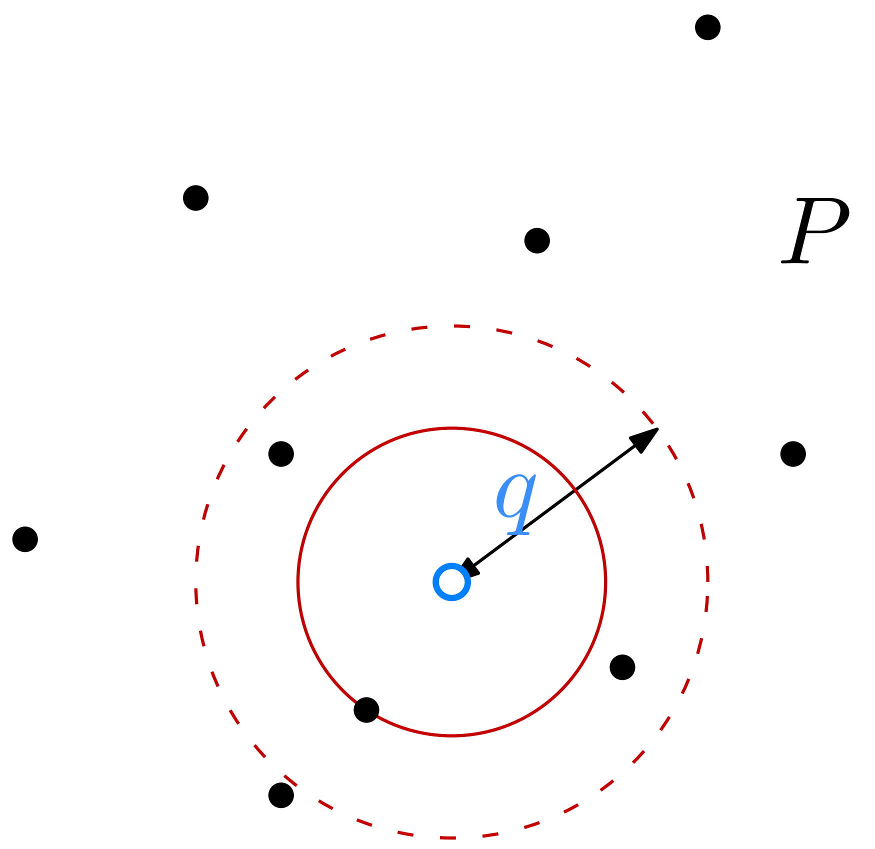

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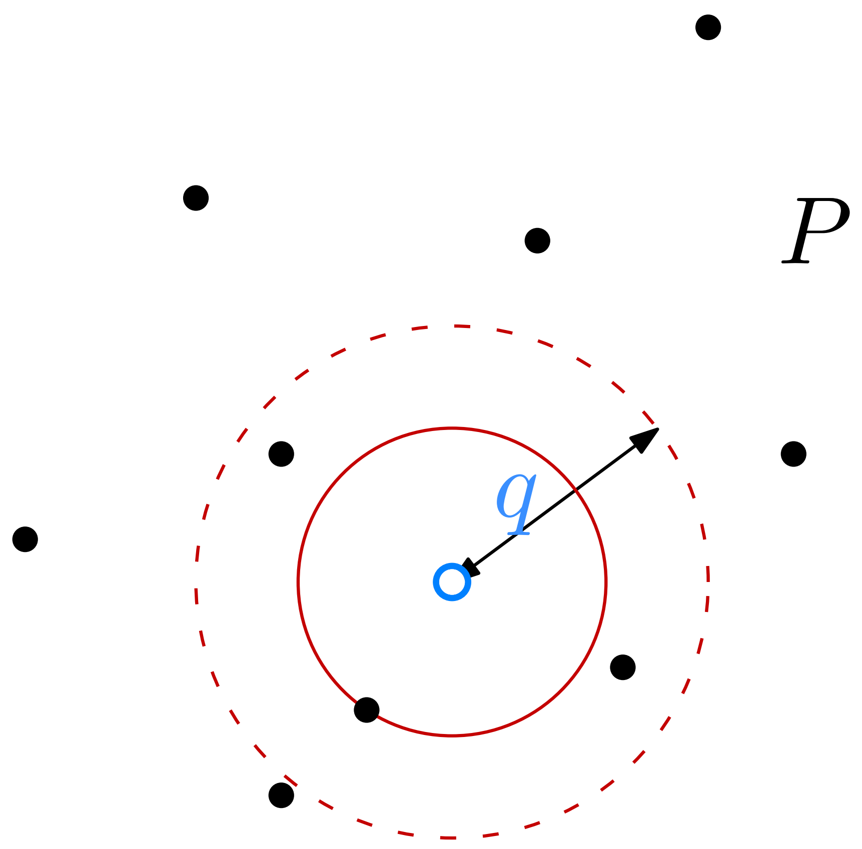

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combined: $(1+\varepsilon)$-ANN in low dimensions

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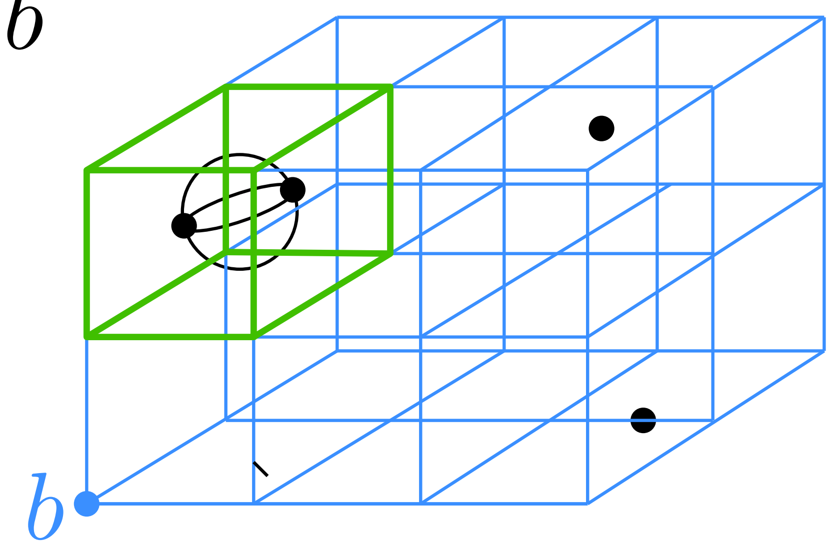
recap: shifting grids and quadtrees by a random vector b

For a ball B of radius r: the probability that B is not in a single cell of $G^d(b, \Delta)$ is at most $\min\left(\frac{2dr}{\Delta}, 1\right)$.



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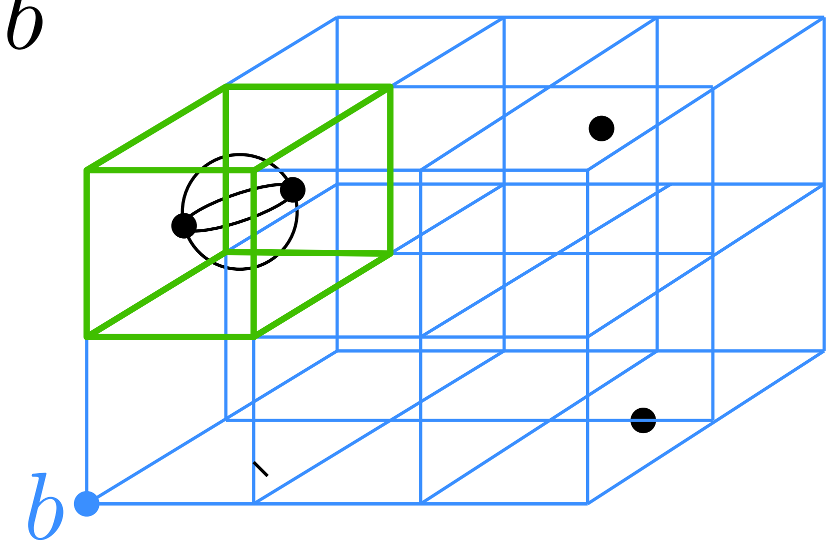
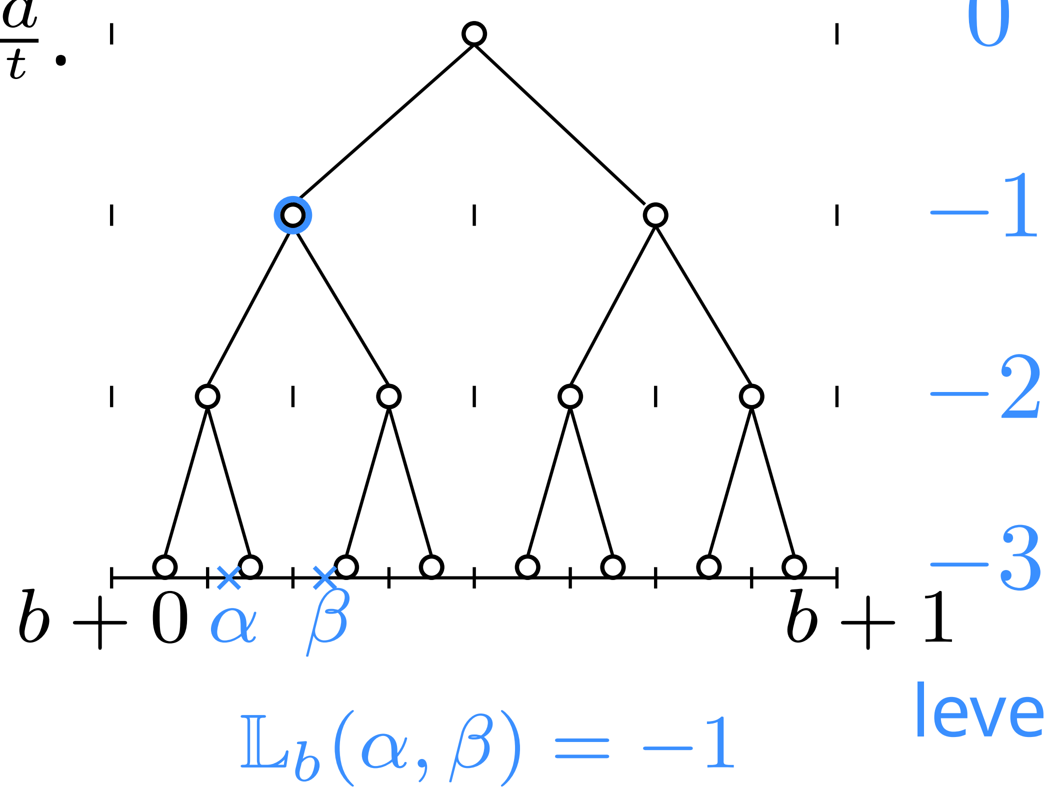


For $t > 0$ holds $\mathbb{P}\left[\mathbb{L}_b(p,q) > \log_2 ||p-q|| + t\right] \le \frac{4d}{2^t}$.

With high probability p and q in same cell at level $\log_2 ||p-q|| + c \log n$ (size of cell: $||p-q|| n^c$)



ANN: today

Approximating a metric space by a hierarchical well-separated tree (HST)

hierachical well-separated trees

simple $(n-1)$-approximation

fast $n^{O(1)}$-approximation in $\mathbb{R}^d$

ANN via point location among balls

simple construction

handling a range of radii

ANN data structure based on HST

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. . . **and next time**

point location among approximate balls

approximate Voronoi diagrams

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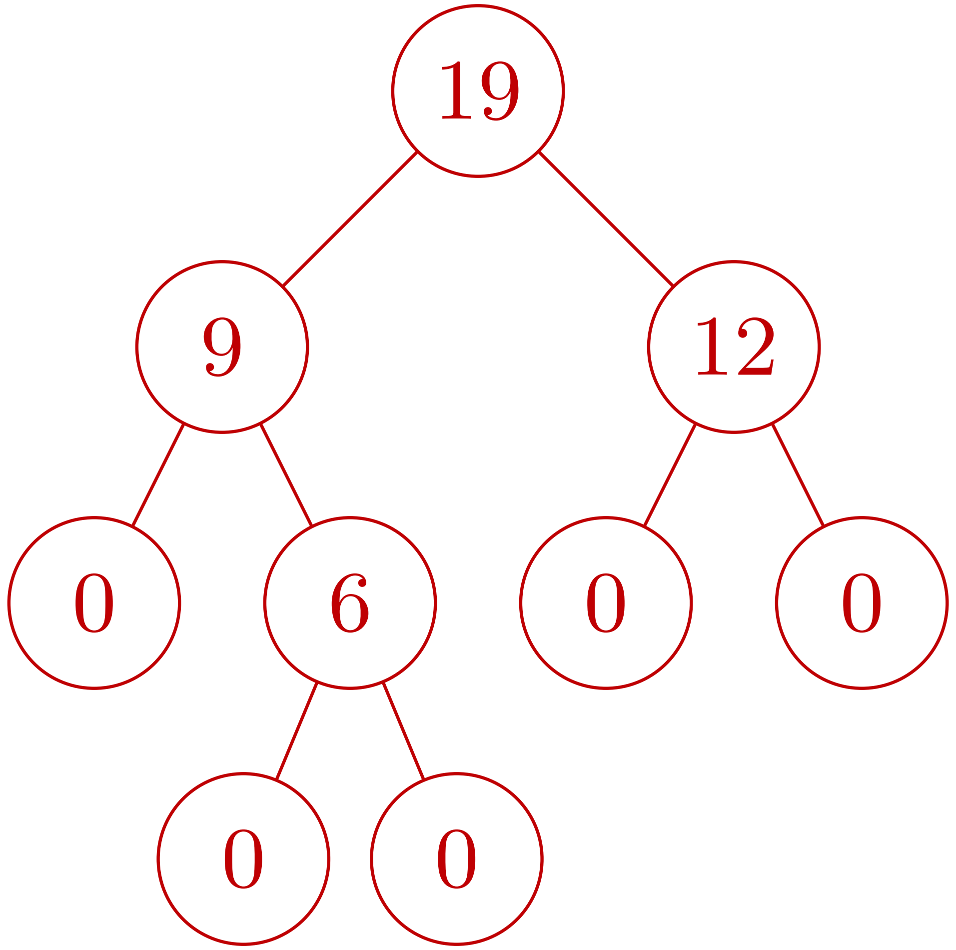

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metric space $M = (X, d)$:

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a distance function $d\colon X \times X \to [0, \infty)$

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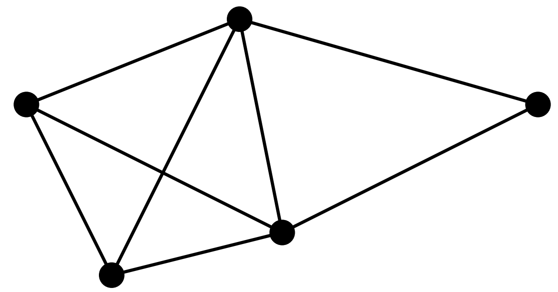

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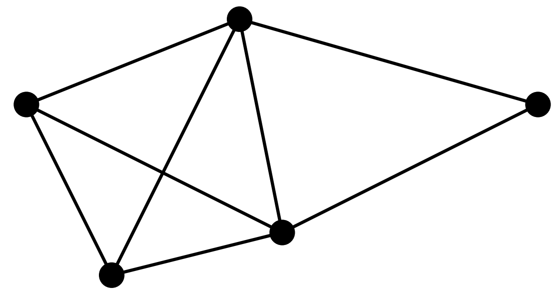

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Question: Examples of metric spaces?

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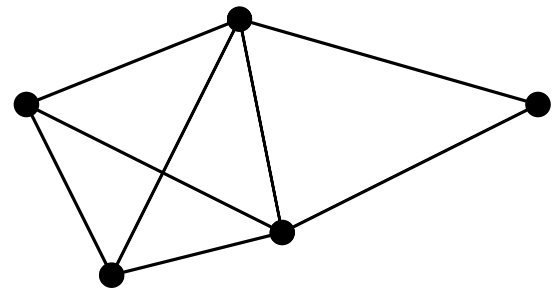

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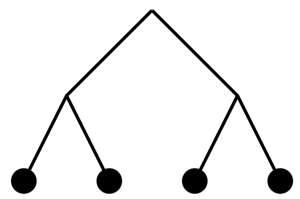
We want:

compact, hierarchical(, approximate) representation



Hierarchically well-separated tree (HST)

HST: rooted (binary) tree T over P with

- label $\Delta_v \geq 0$ for each node $v \in T$
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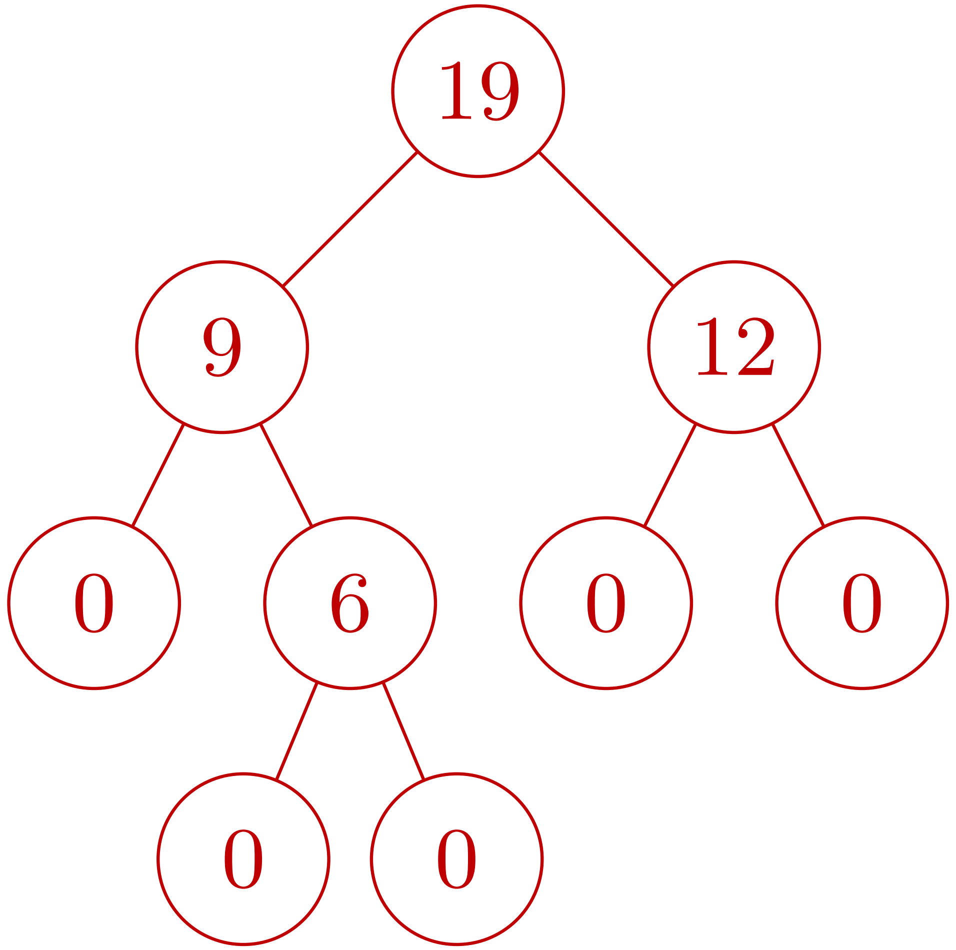

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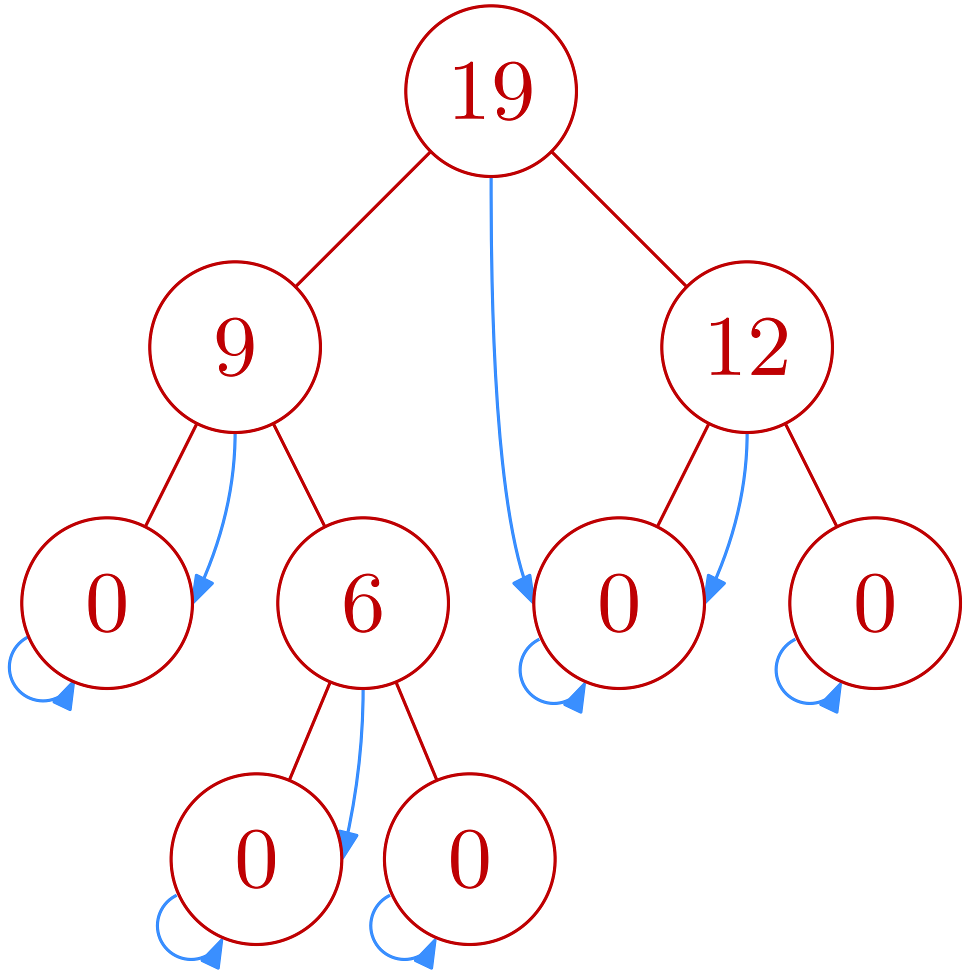

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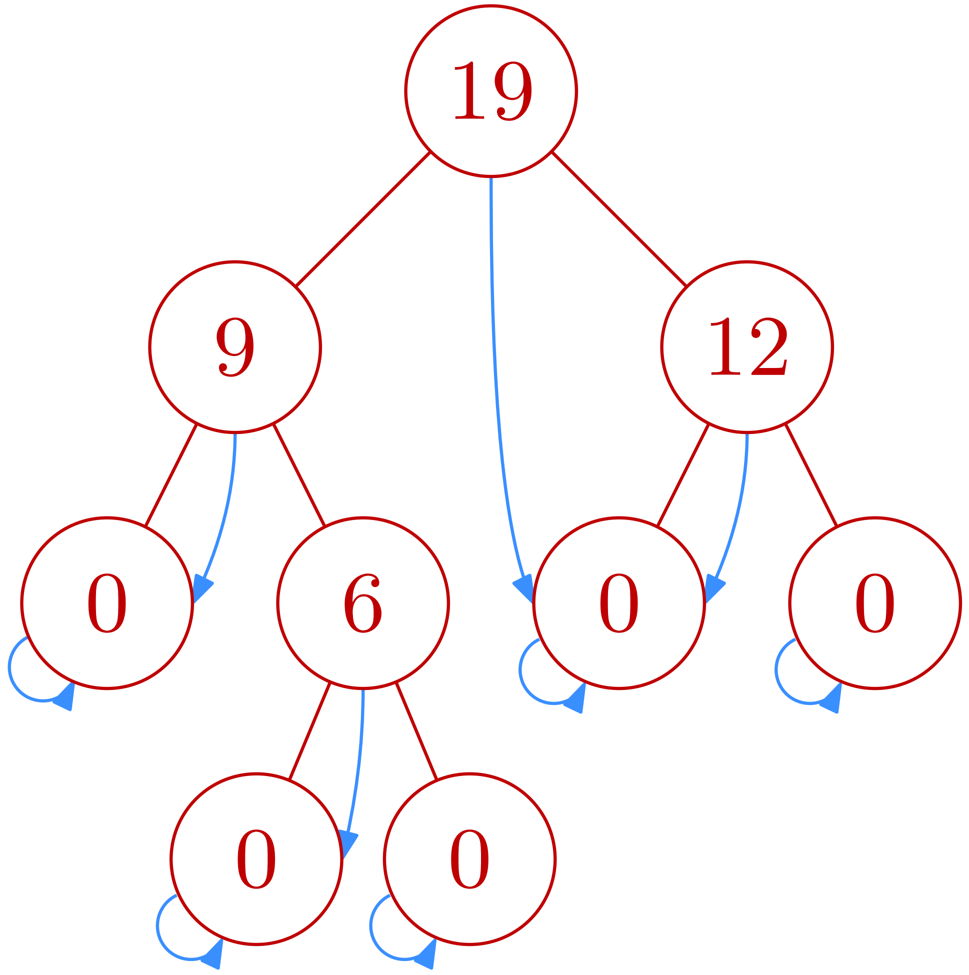
example: quadtree with $\Delta_v =$ diameter of cell



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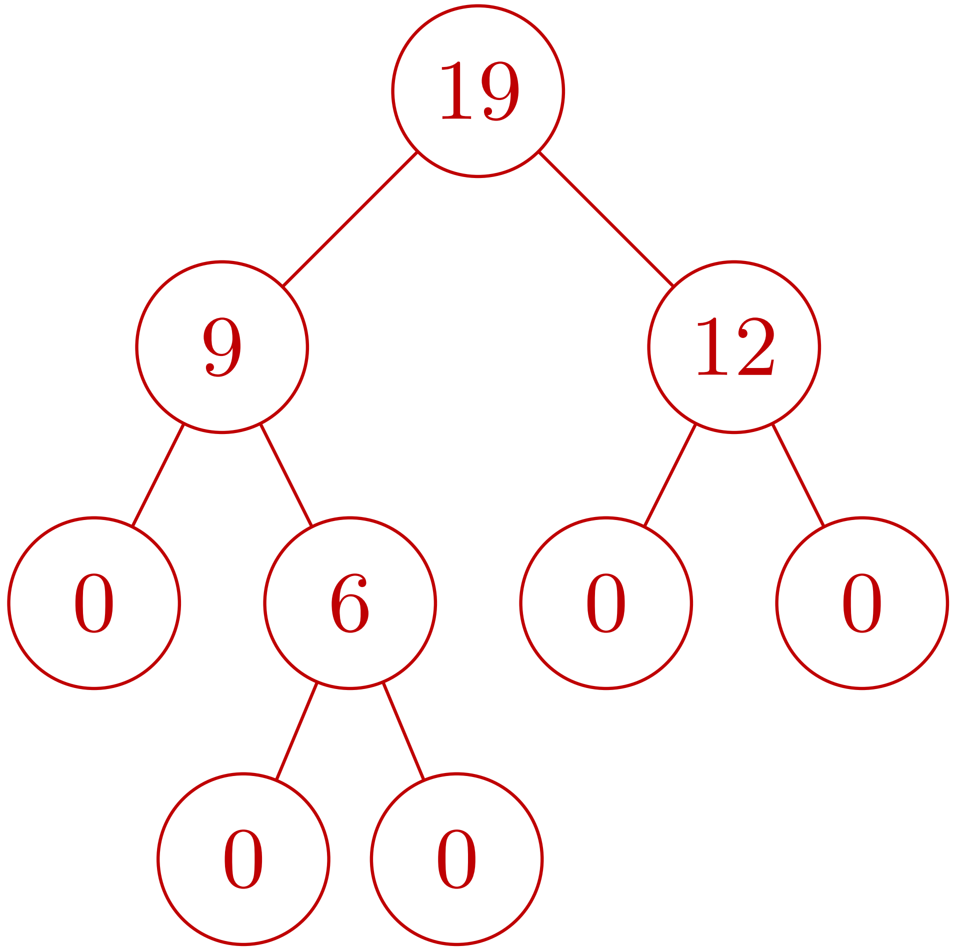

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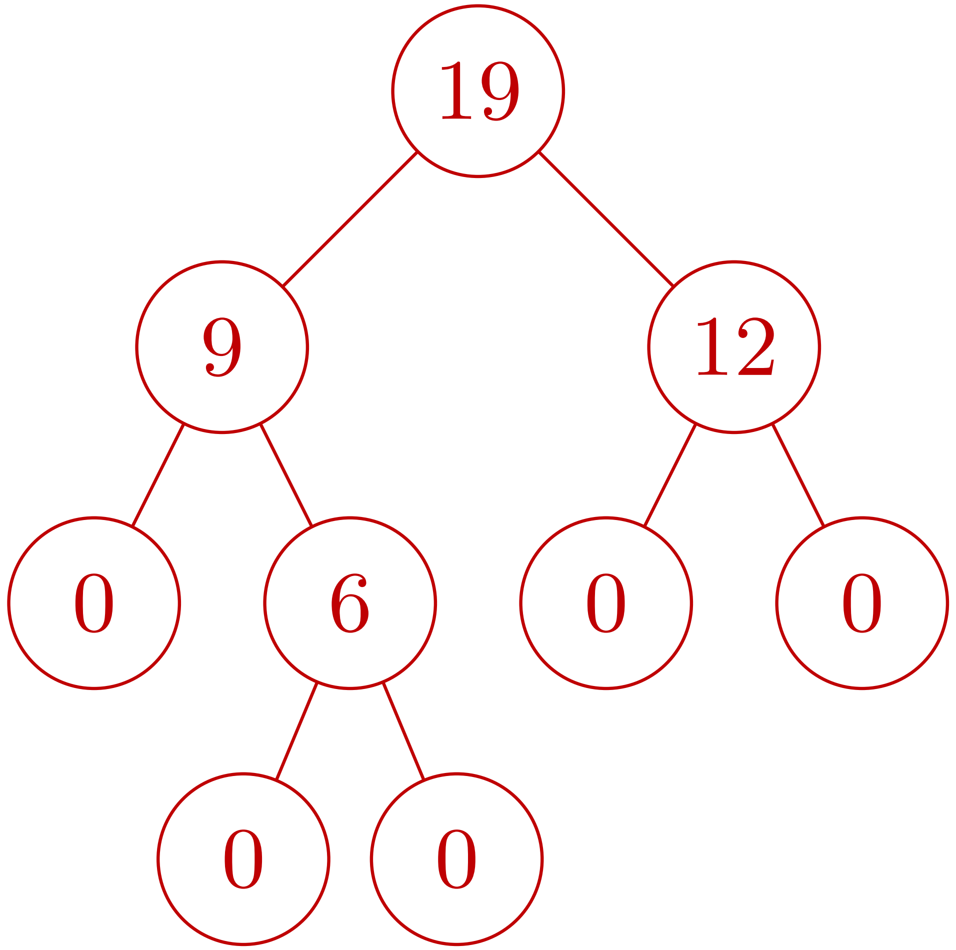

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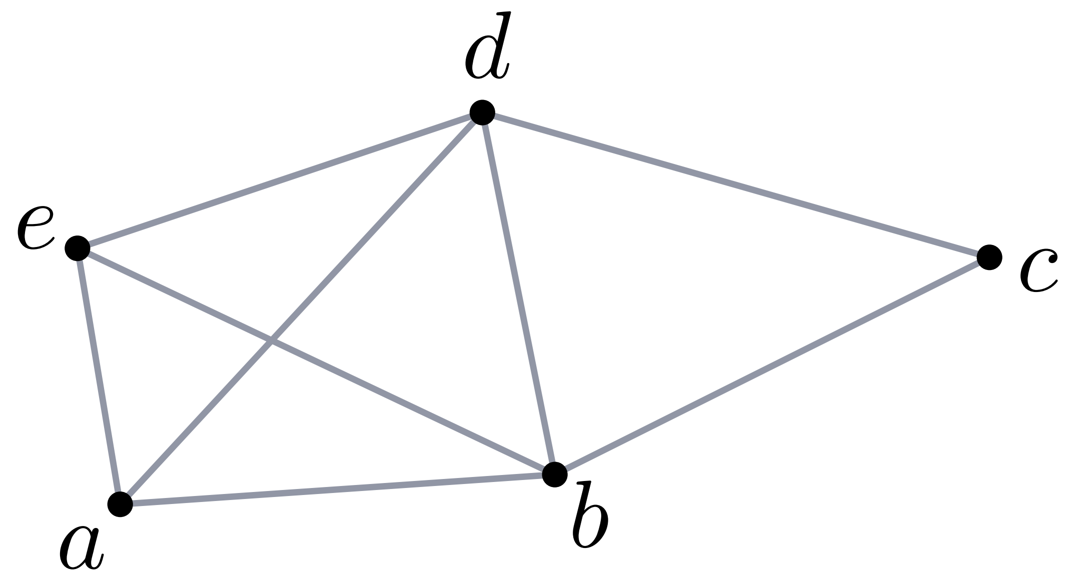

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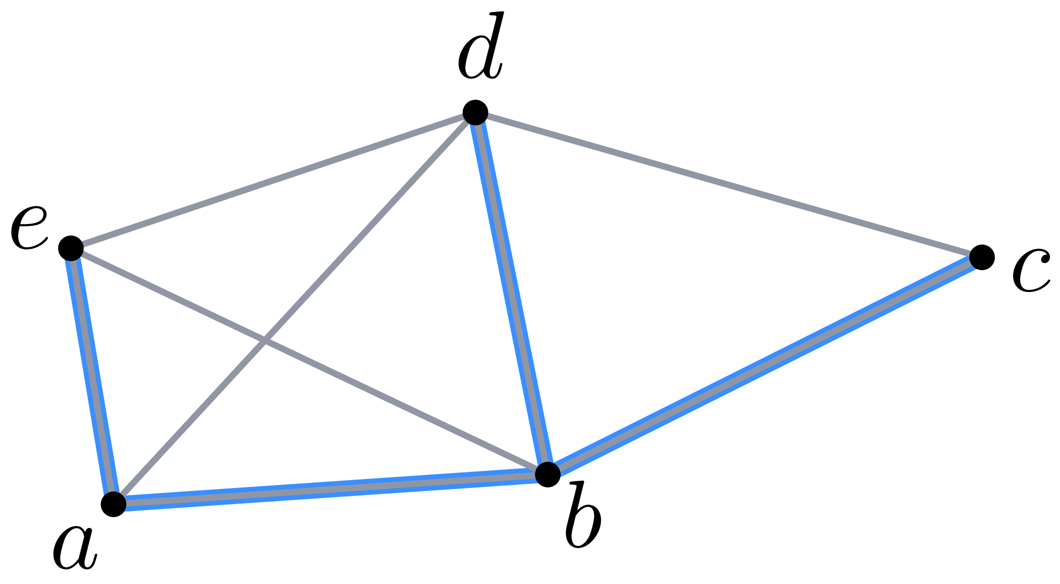

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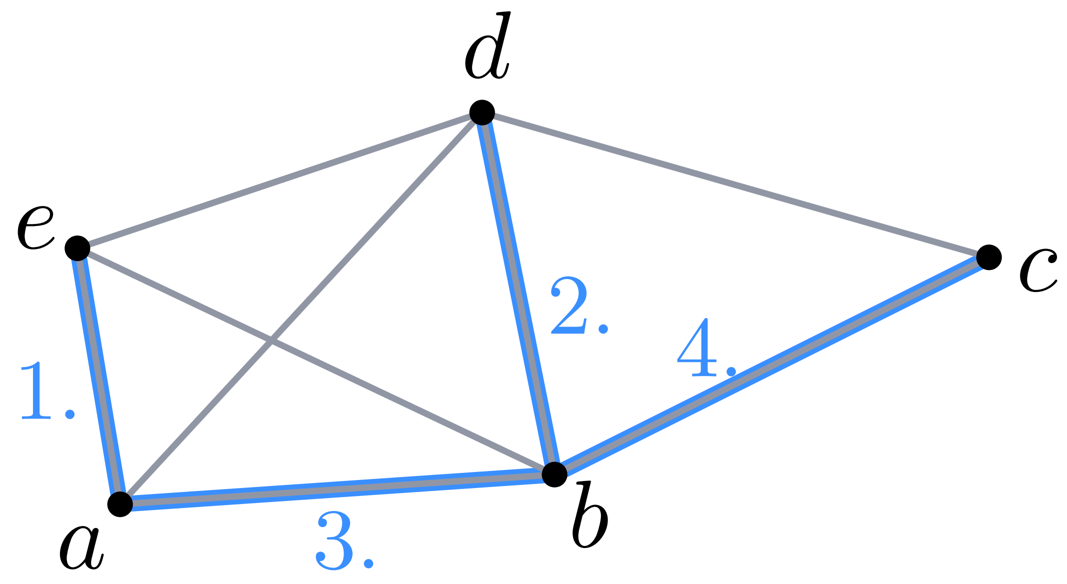

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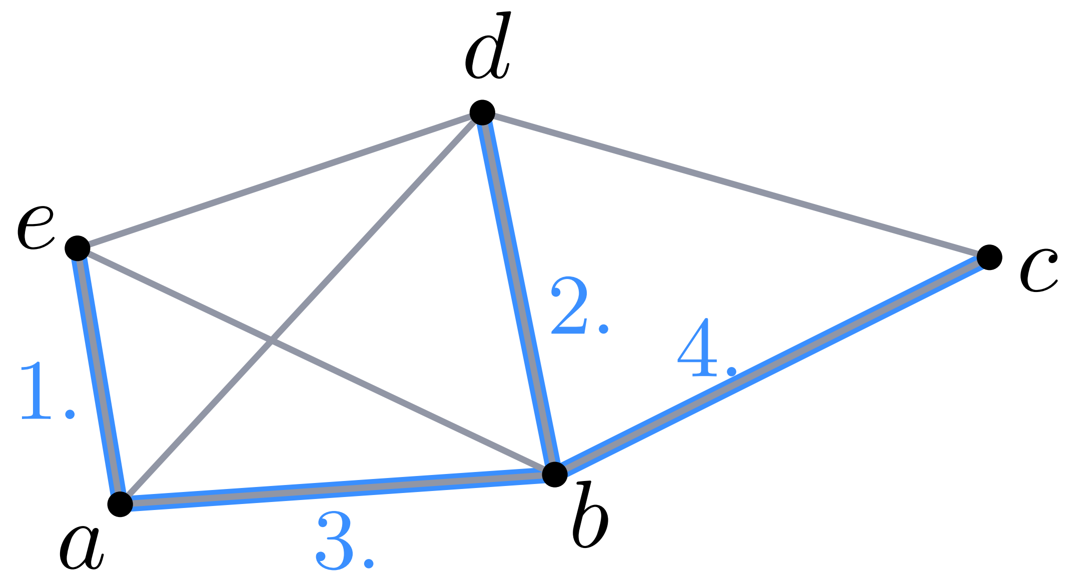

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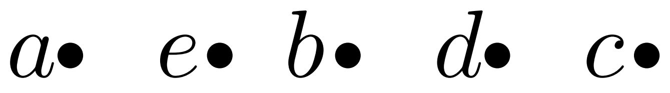

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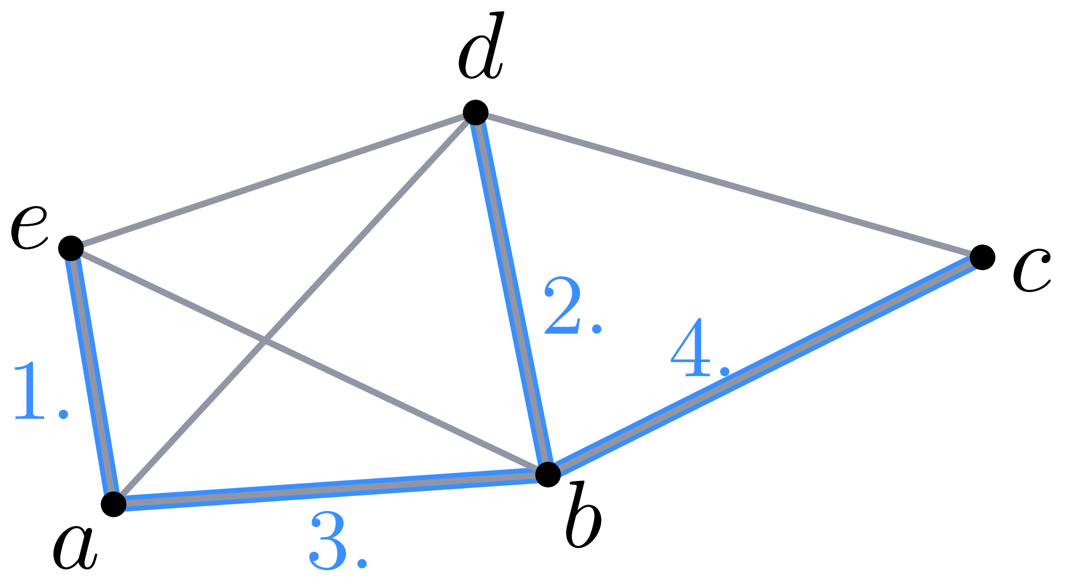

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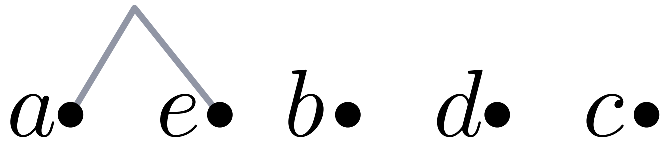

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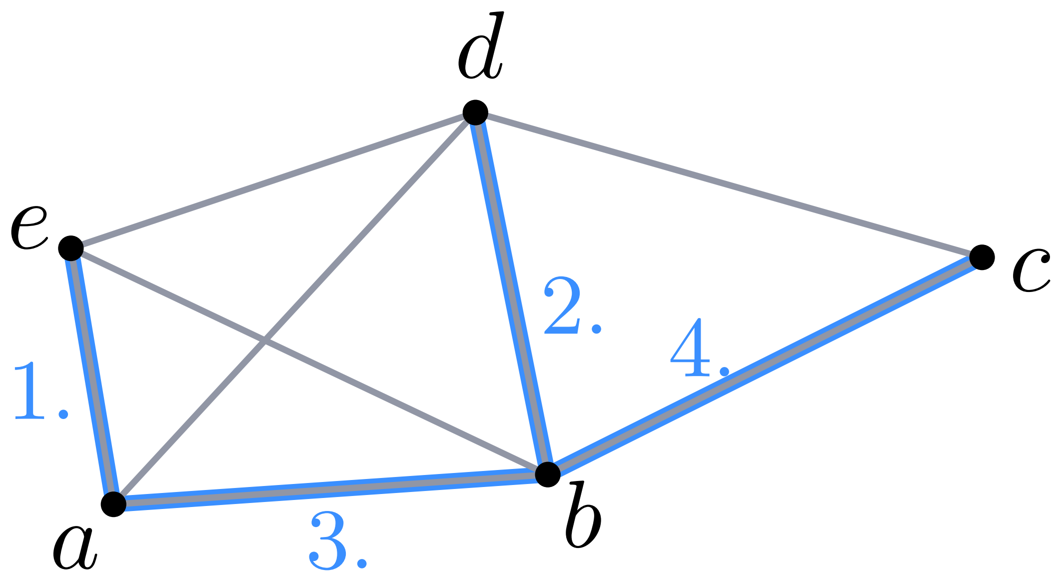

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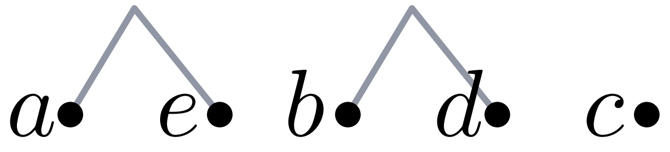

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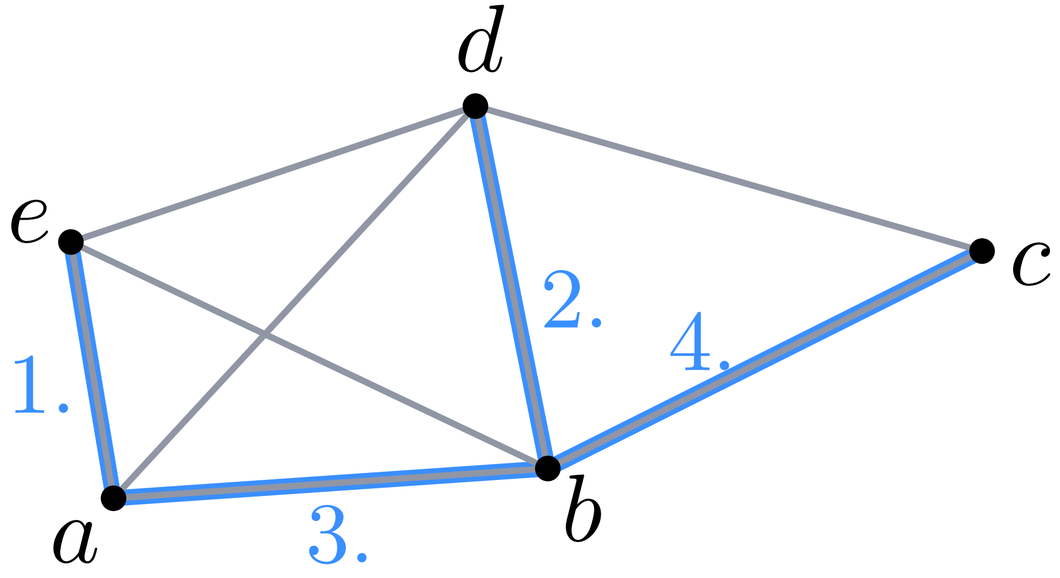

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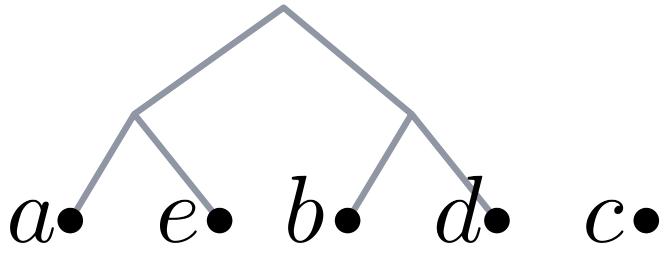

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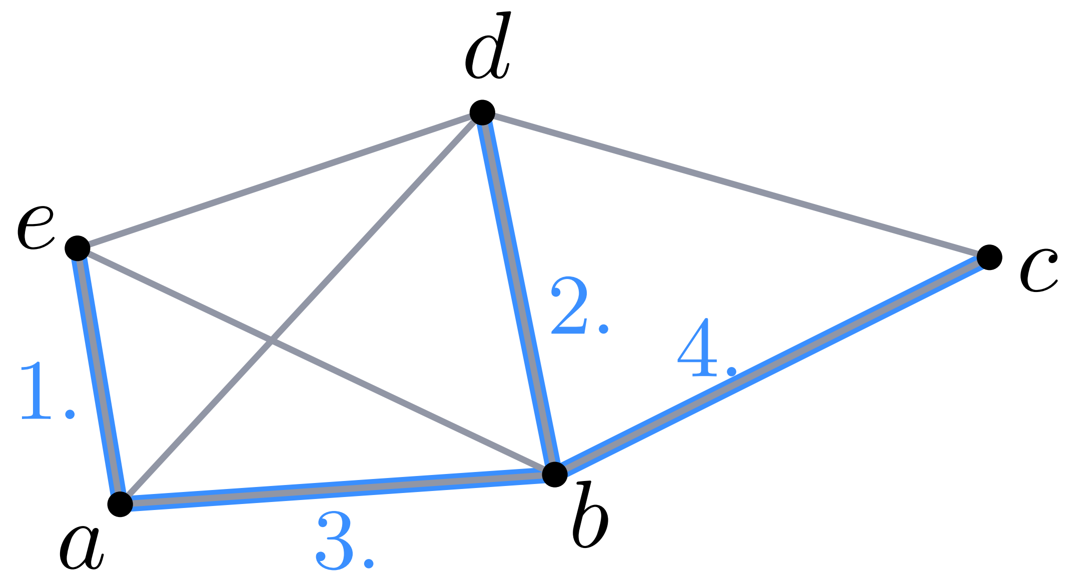

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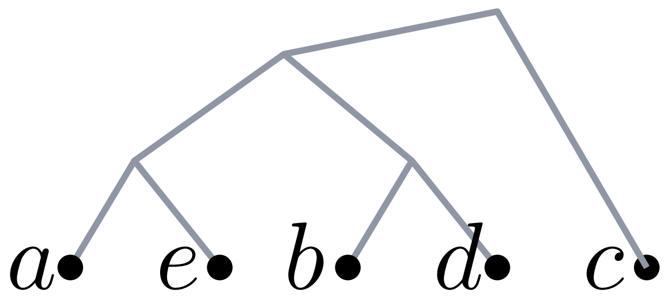

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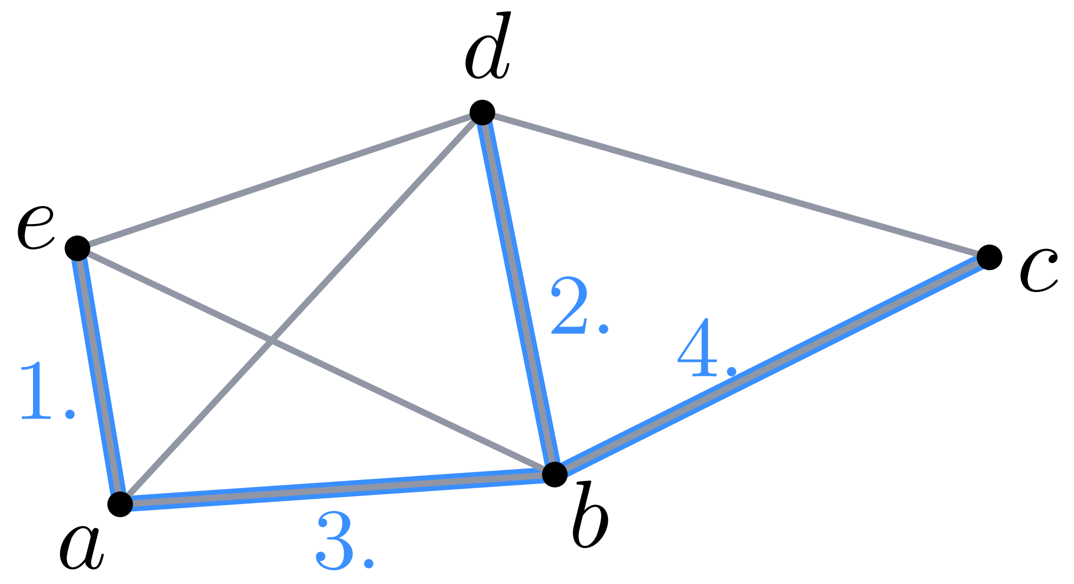

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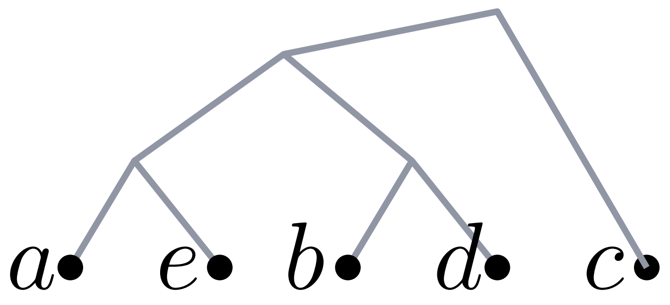

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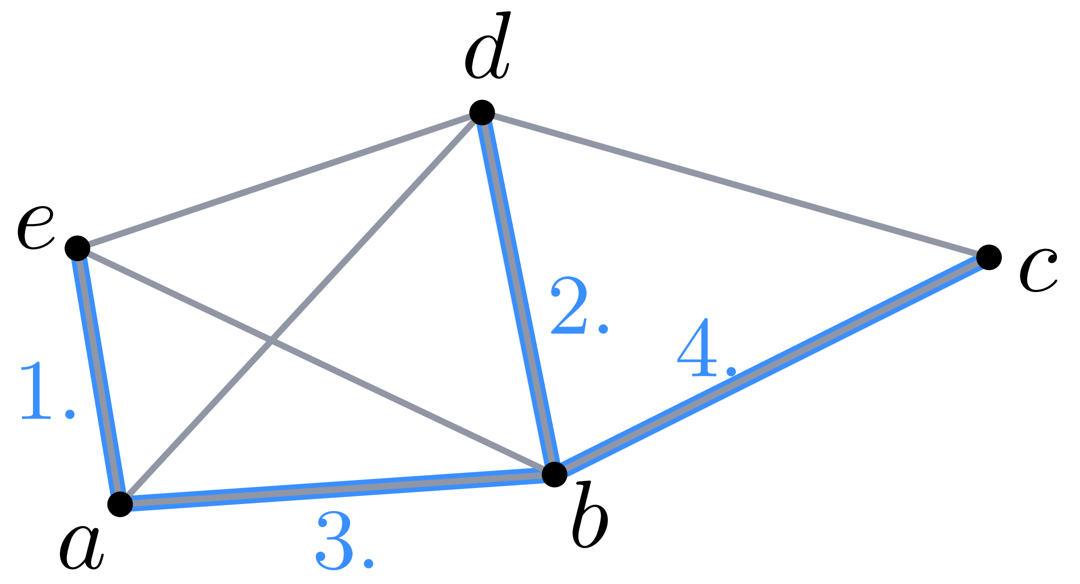

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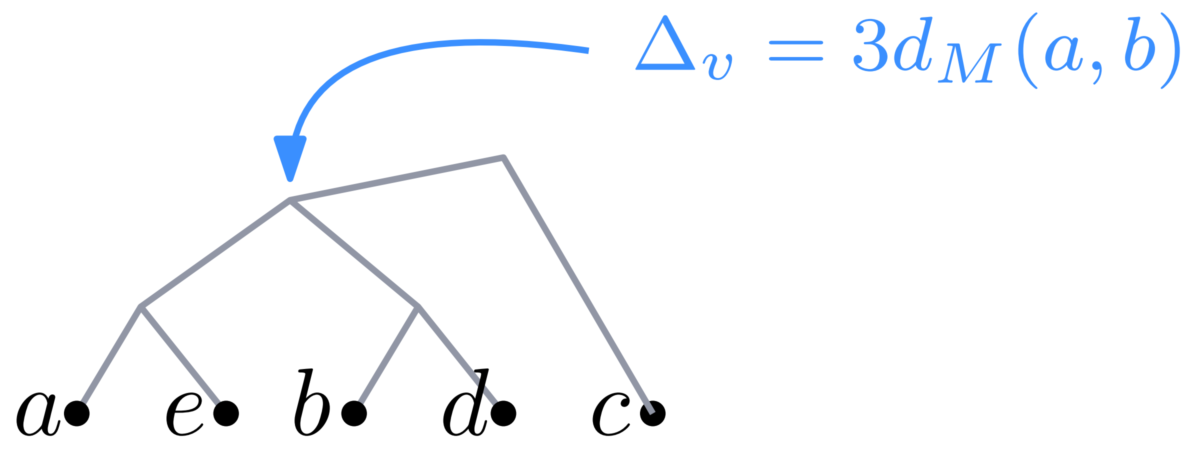

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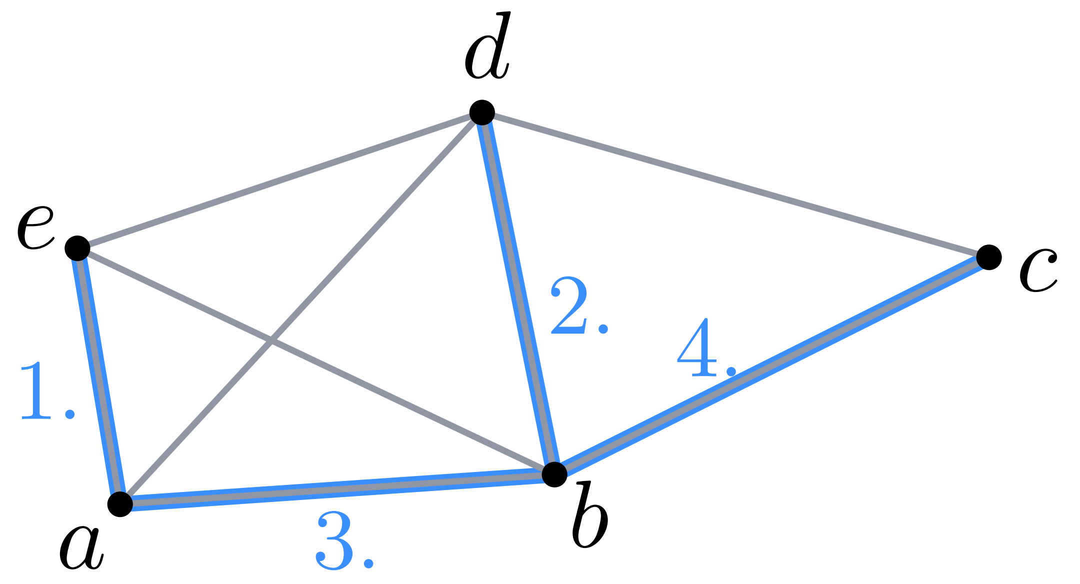
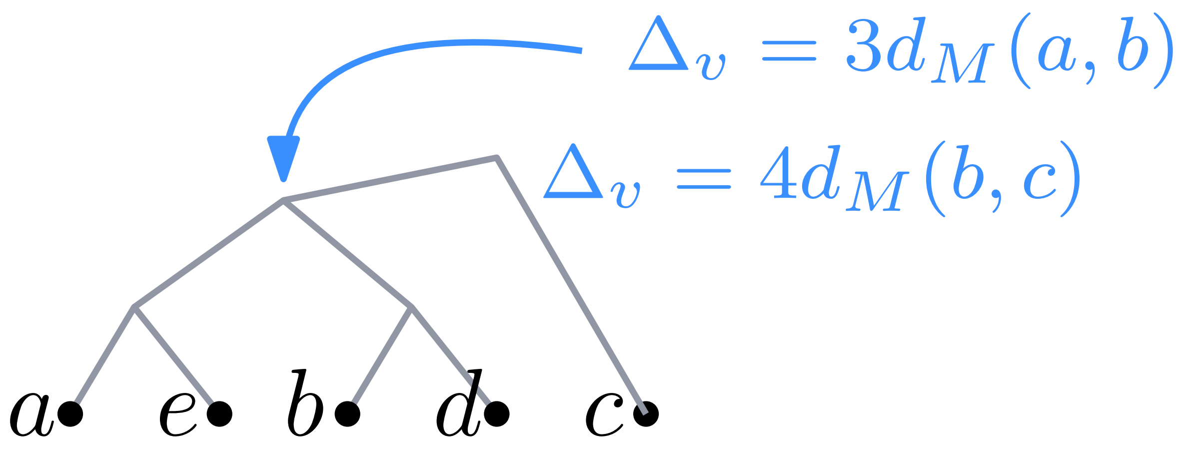


simple HST construction:

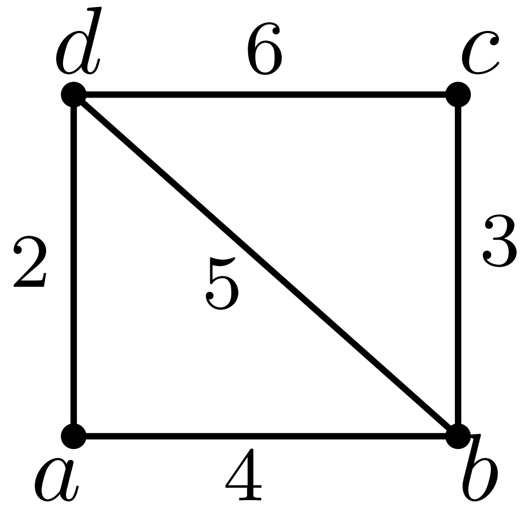
given: metric $M = (P, d)$ as weighted graph

1. compute minimum spanning tree
2. sort edges of MST from short to long
3. build HST bottom-up: add vertex for MST edge with the two merged components as children
4. set Δ_v = 0 for leaves
5. for v with point set P_v in subtree and MST edge weight w: $\Delta_v := (|P_v| - 1)\, w$



Small Assignment

Construct the HST for this metric space M (weights shown on edges). Is it a t-approximation of M?



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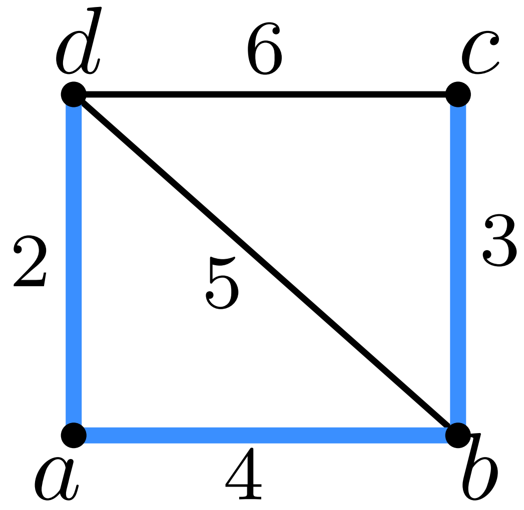

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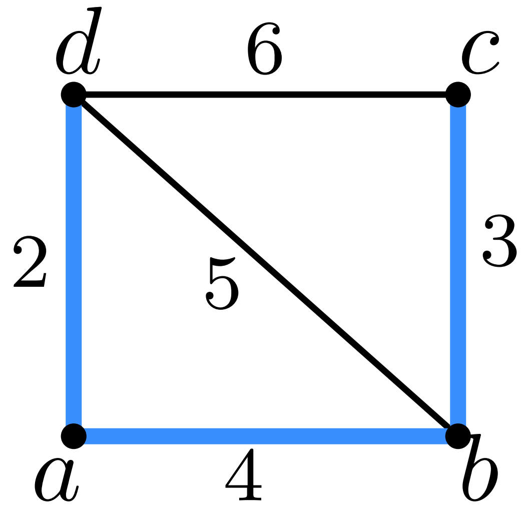

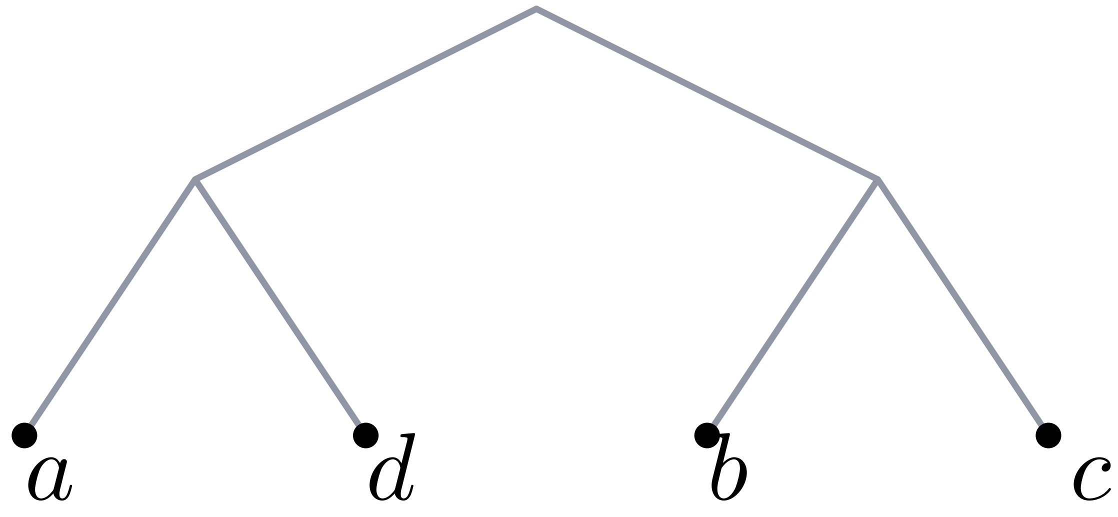

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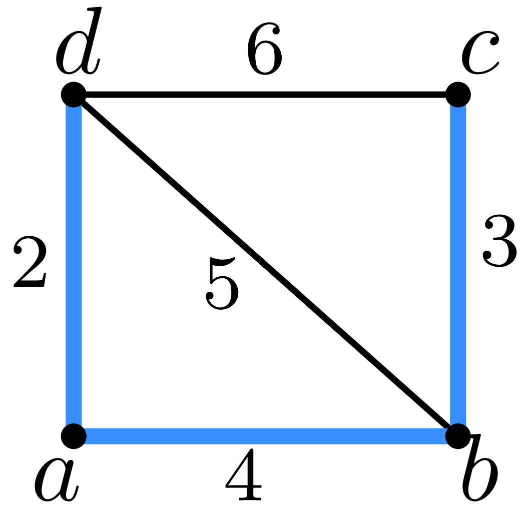

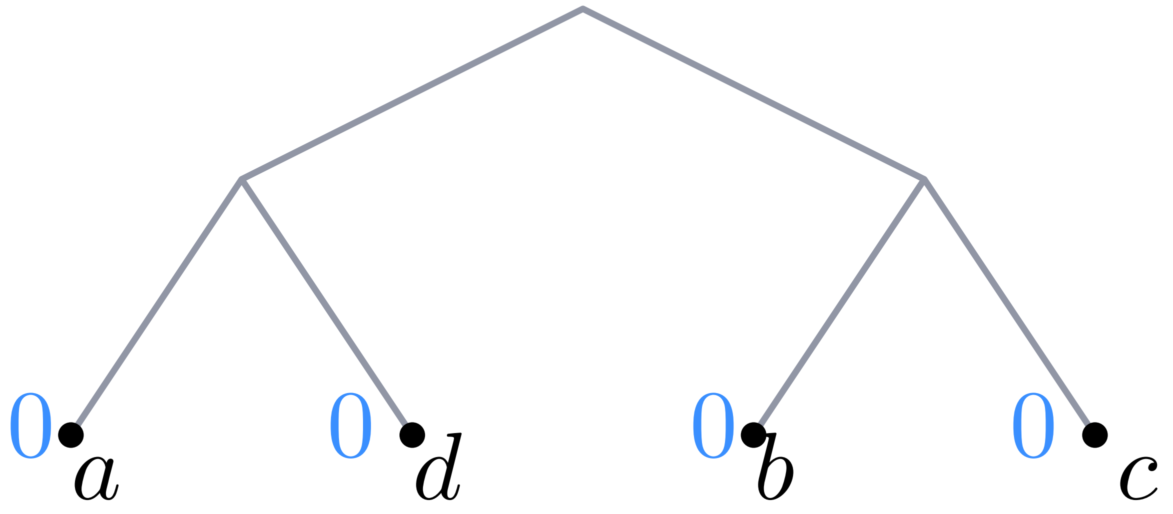

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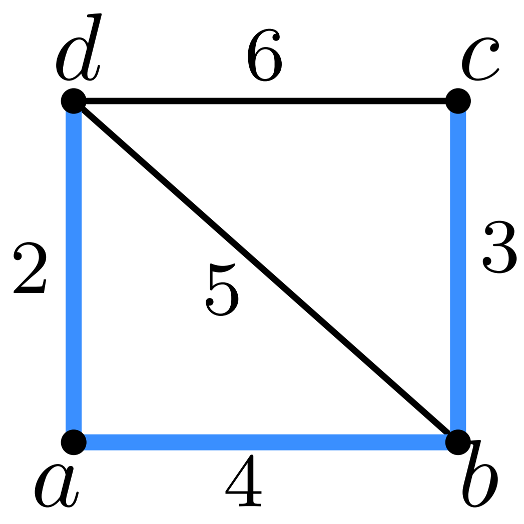

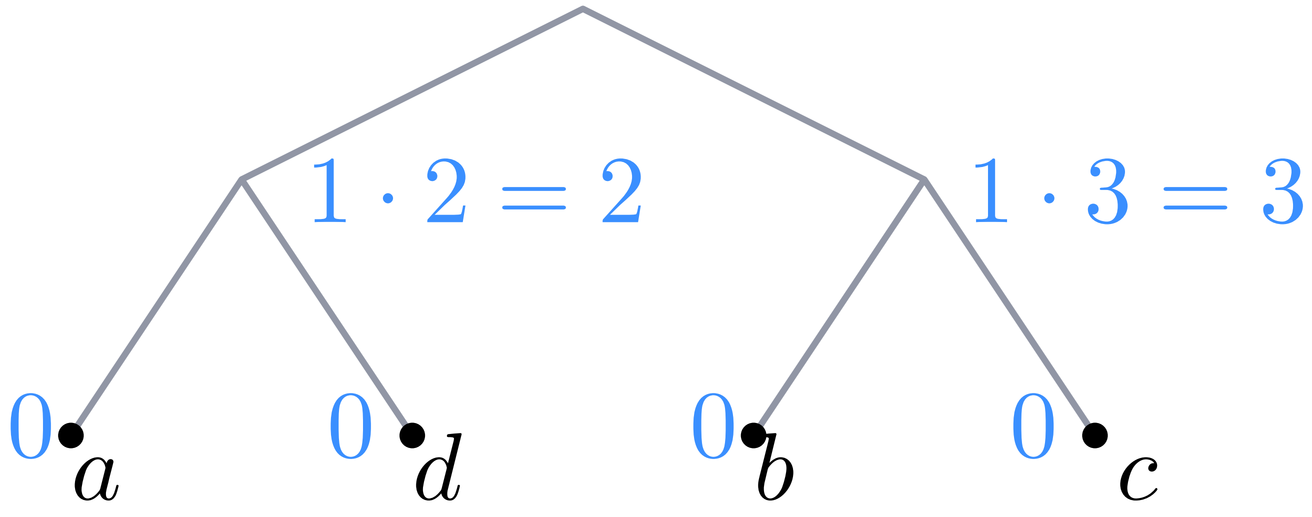

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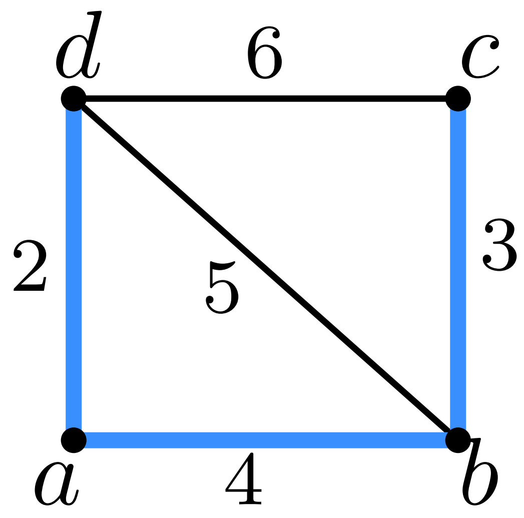

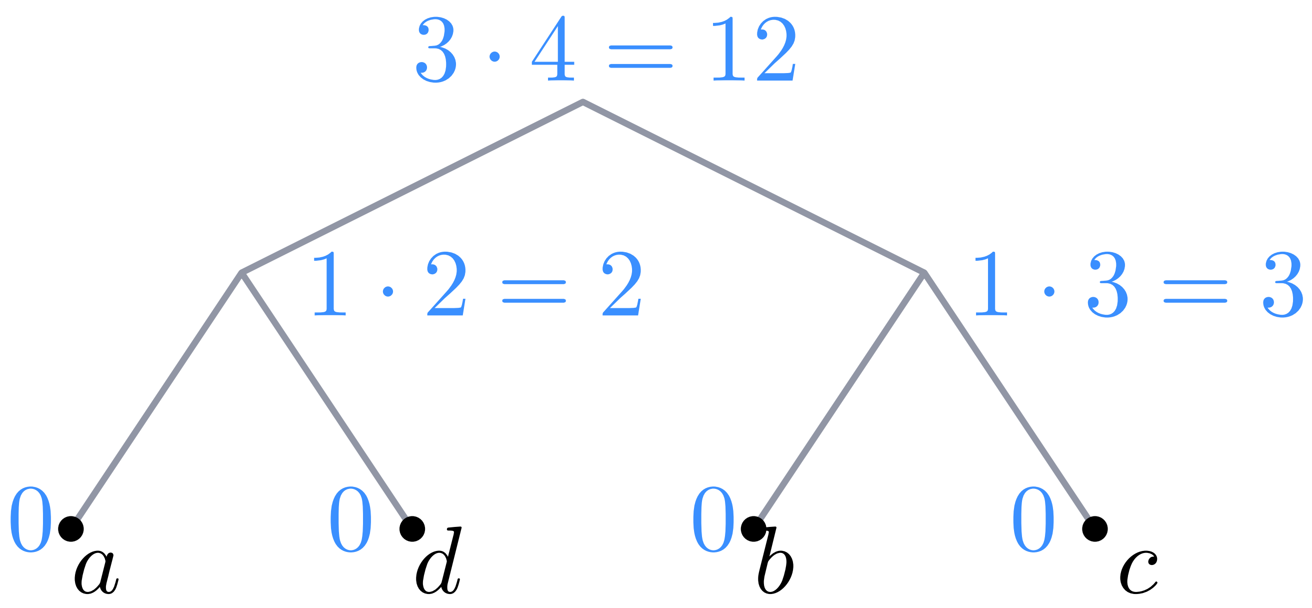

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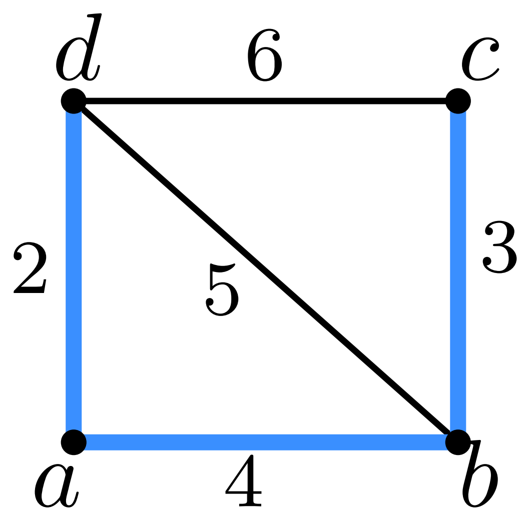
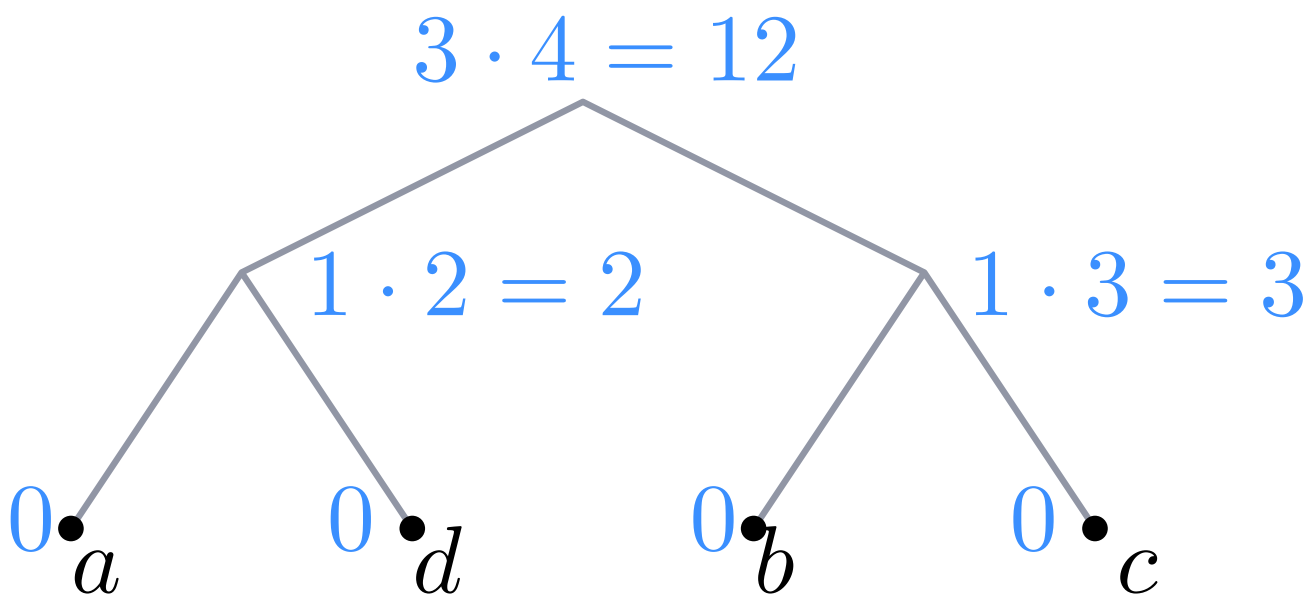


$$d_M: \begin{bmatrix} 0 & 4 & 7 & 2 \\ 4 & 0 & 3 & 5 \\ 7 & 3 & 0 & 6 \\ 2 & 5 & 6 & 0 \end{bmatrix} \qquad d_{HST}: \begin{bmatrix} 0 & 12 & 12 & 2 \\ 12 & 0 & 3 & 12 \\ 12 & 3 & 0 & 12 \\ 2 & 12 & 12 & 0 \end{bmatrix}$$



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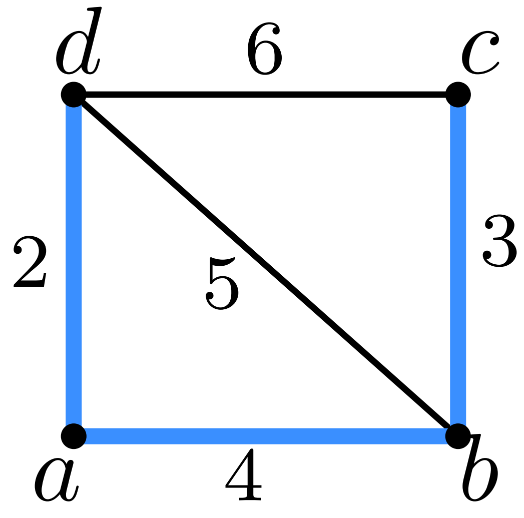

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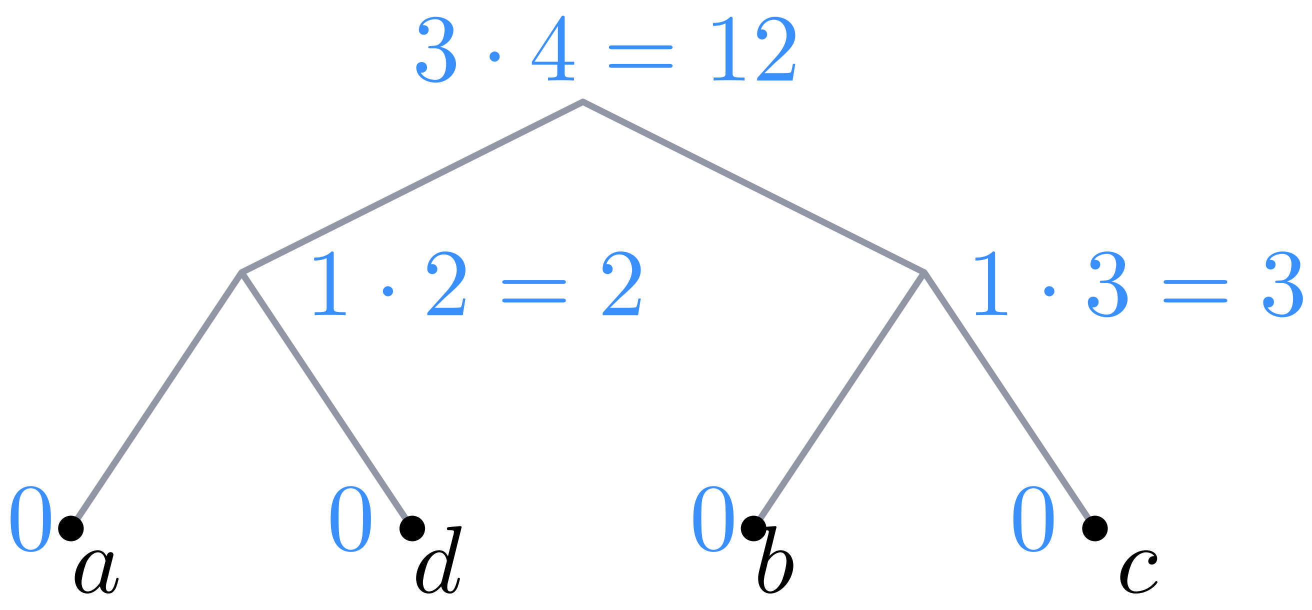
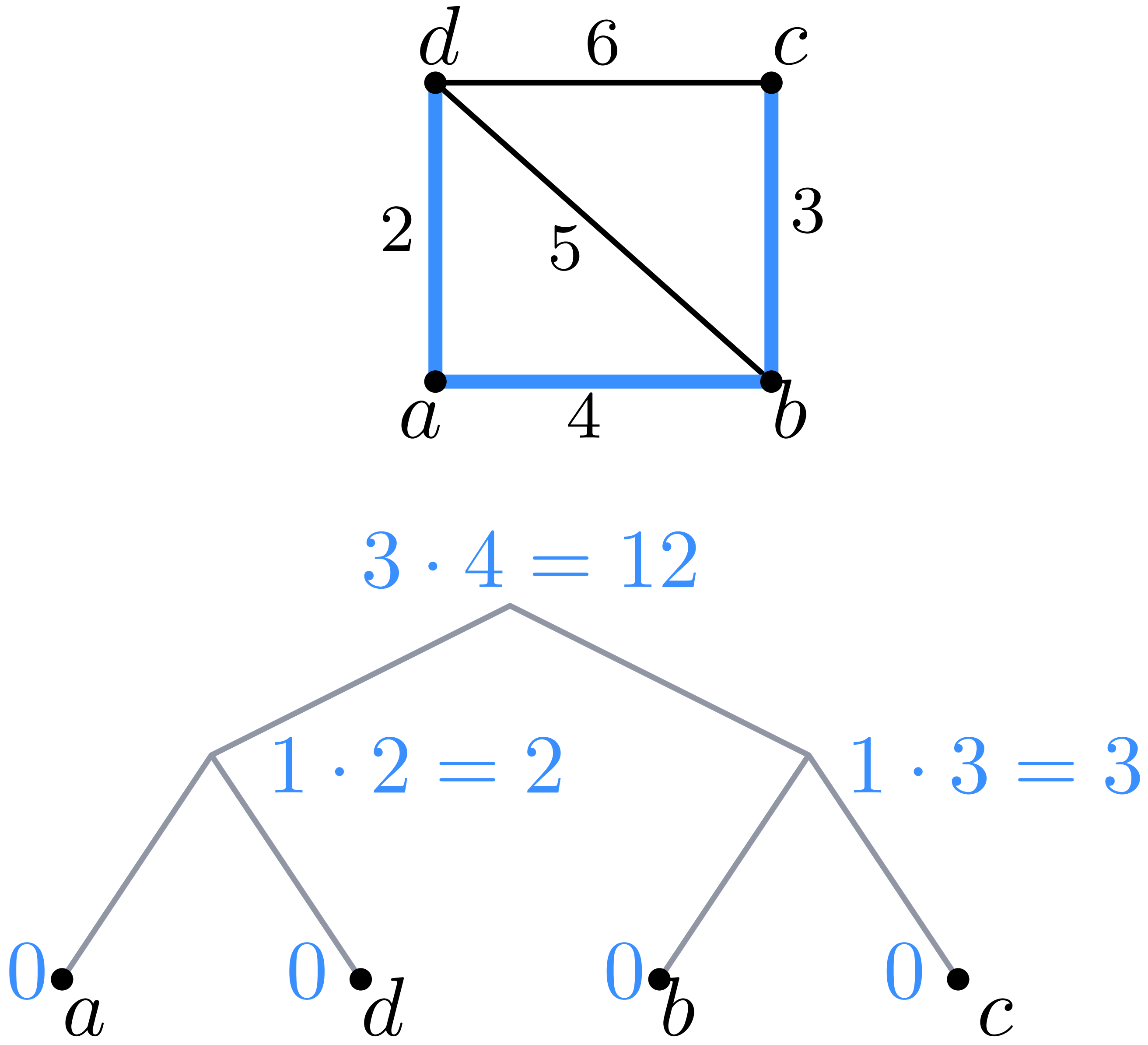


4-approximation

$$\Delta_v := (|P_v| - 1)\, w$$

$$d_M \leq d_{HST} \leq (n-1)d_M$$

(0. if u child of v: $\Delta_u \leq \Delta_v$)



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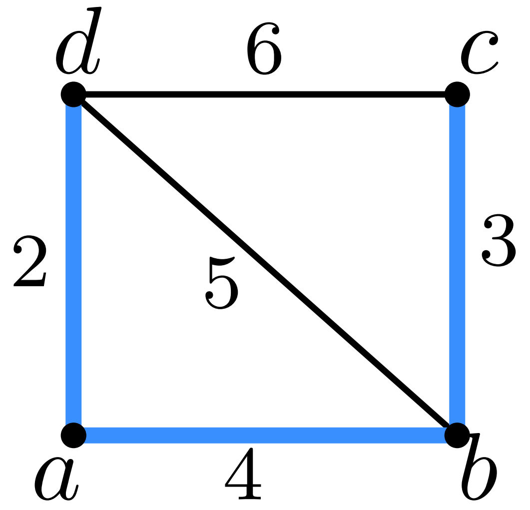
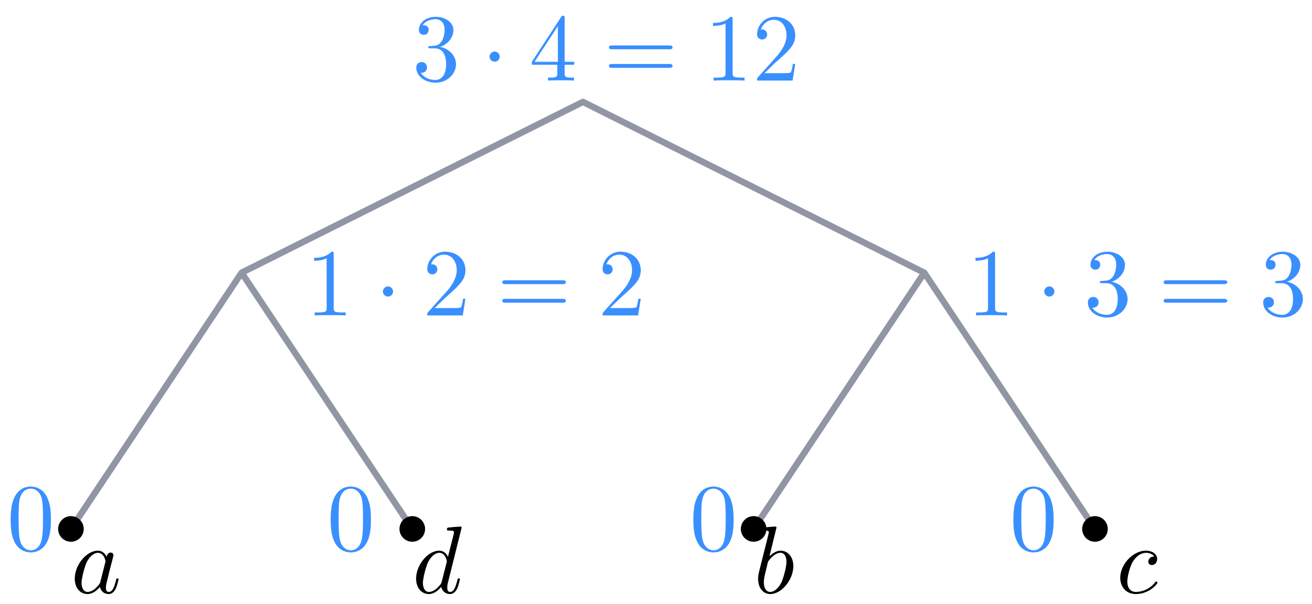
(0. if u child of v: $\Delta_u \leq \Delta_v$)

a.) $|P_u| < |P_v|$

b.) edges handled in increasing order by weight





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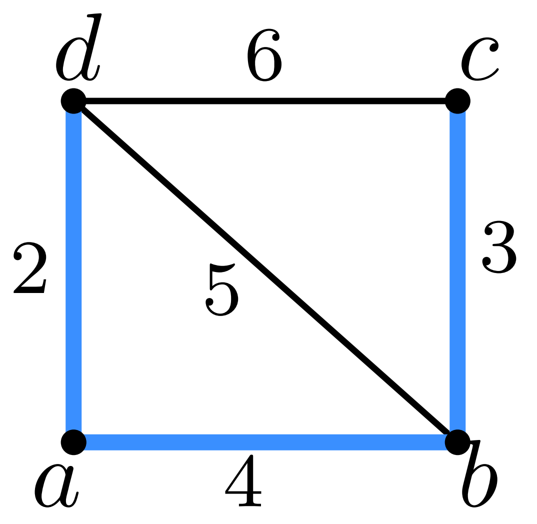

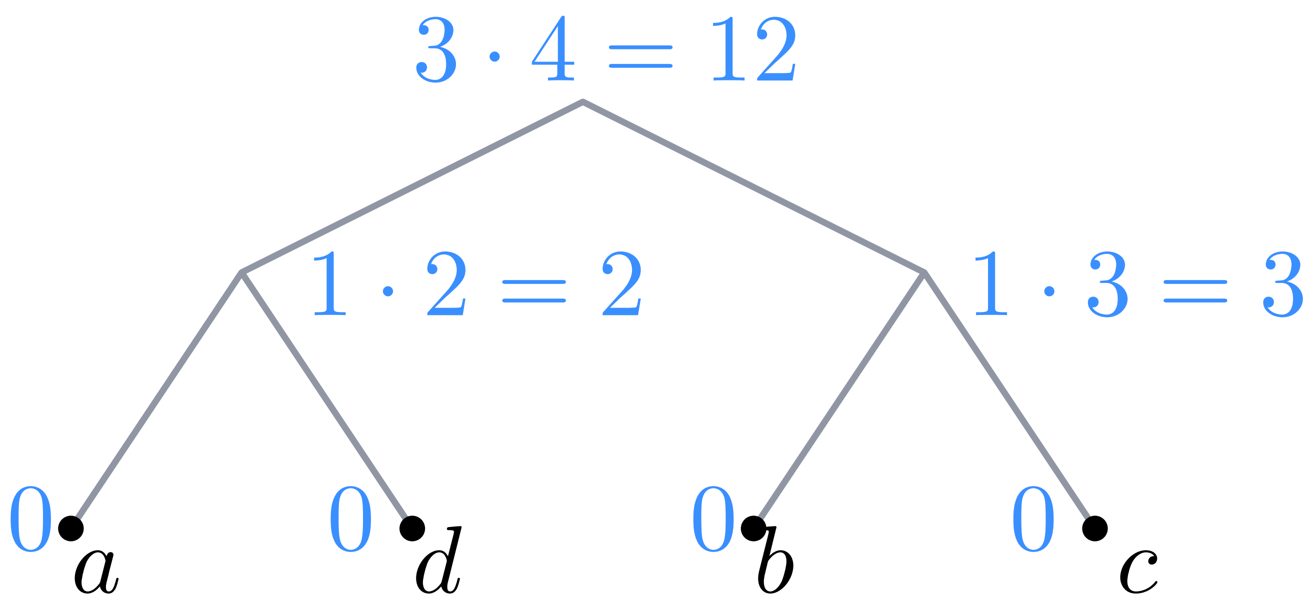

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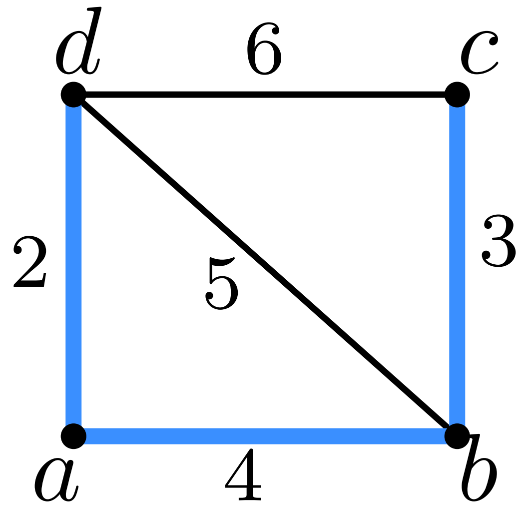
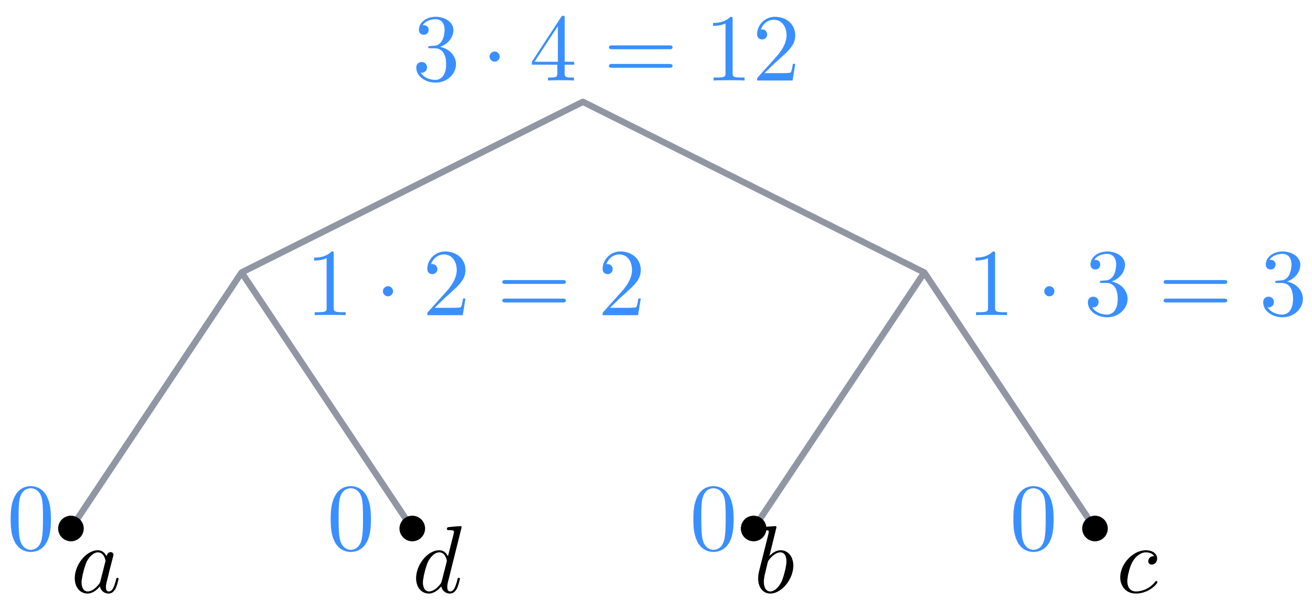
1. $d_{HST} \le (n-1)d_M$

$x, y \in P$, $v := lca(u_x, u_y)$. Then

$|P_v| - 1 \le n - 1$

weight(MST edge for v) $\le d_M(x, y)$





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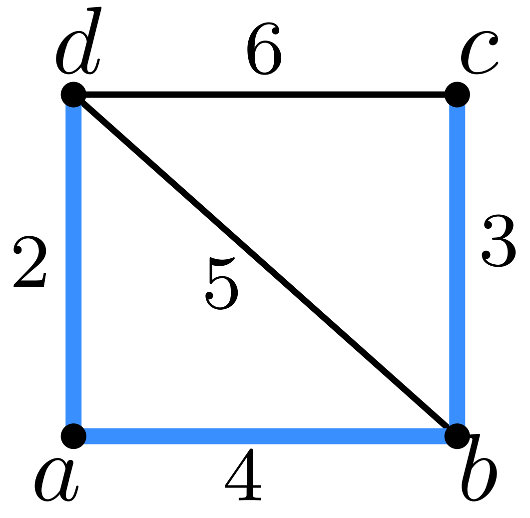
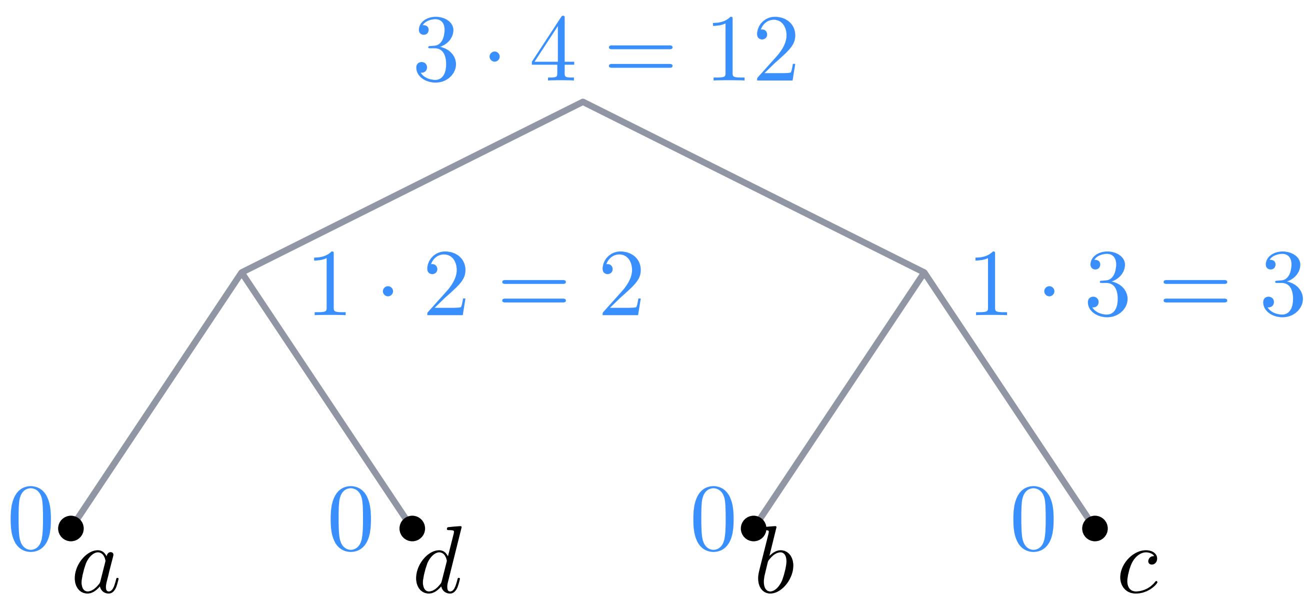
$\Rightarrow d_{HST}(x, y) = \Delta_v \leq (n-1)d_M(x, y)$





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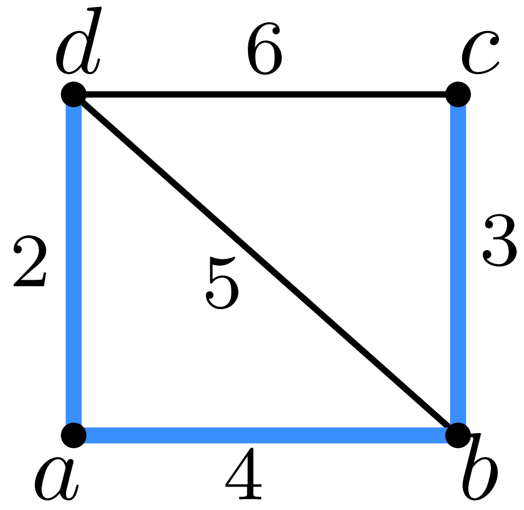
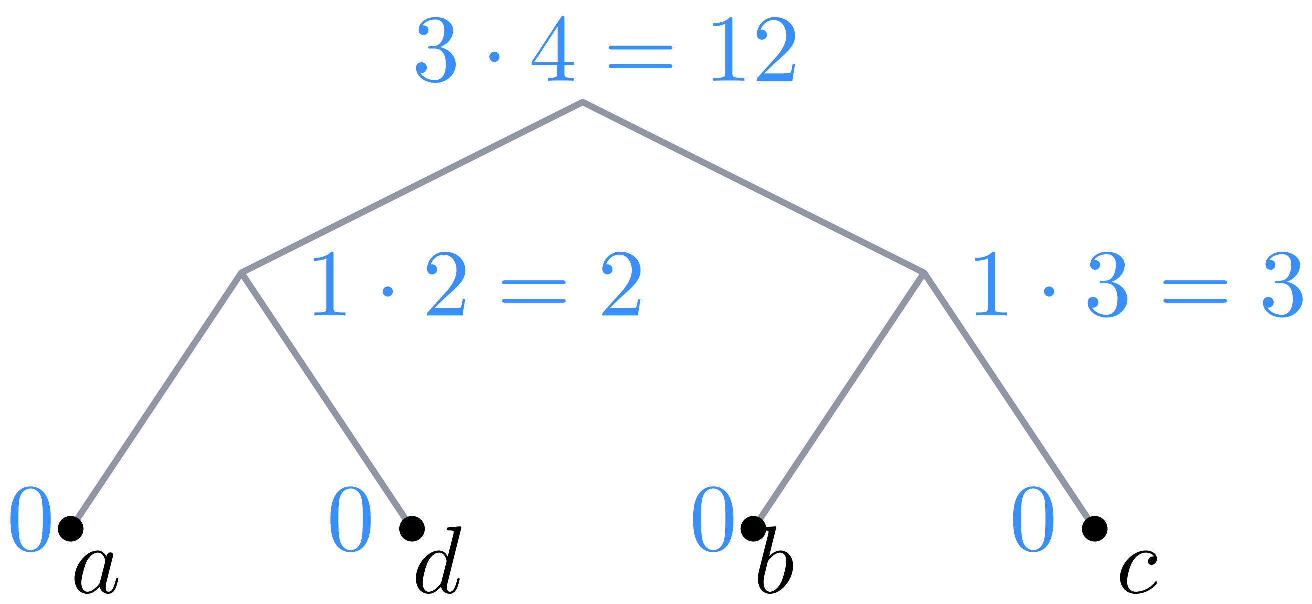
1. $d_{HST} \le (n-1)d_M$
2. $d_M \le d_{HST}$





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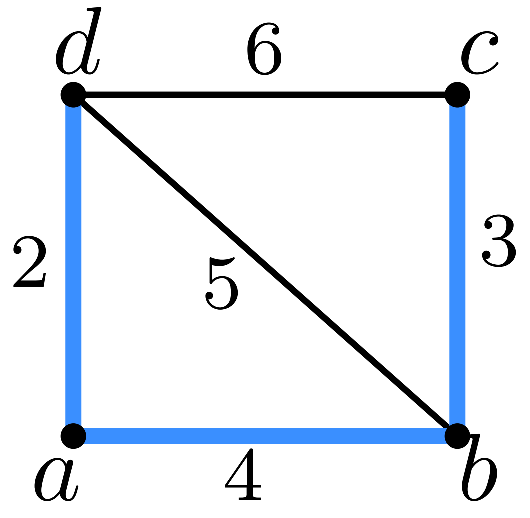

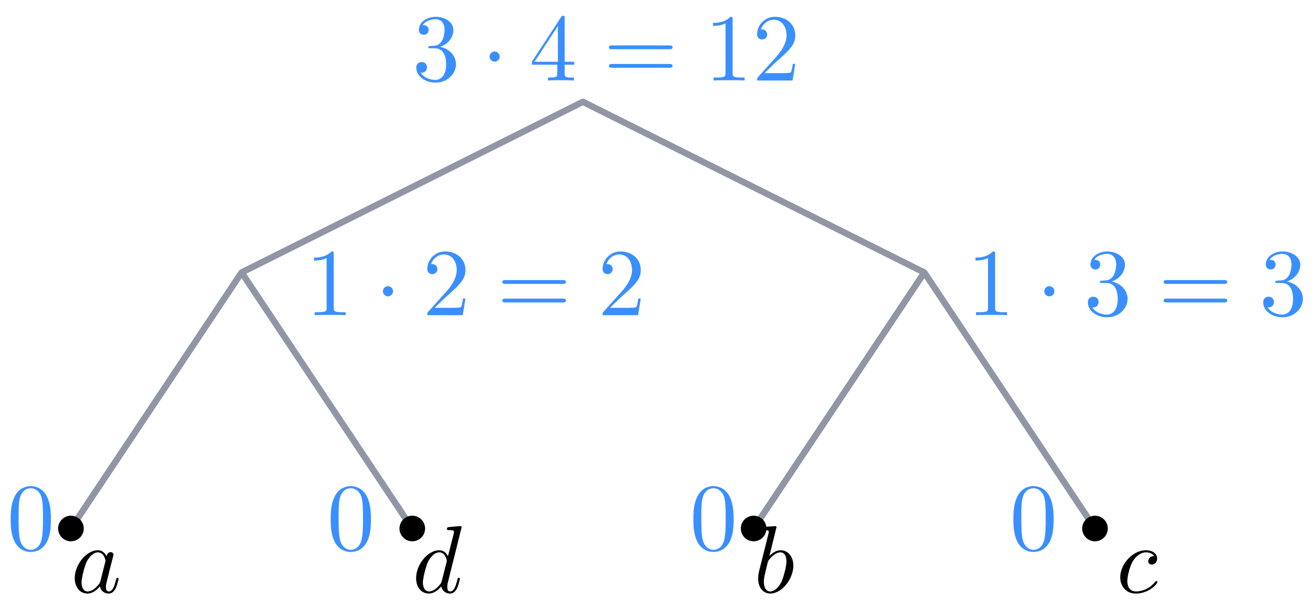

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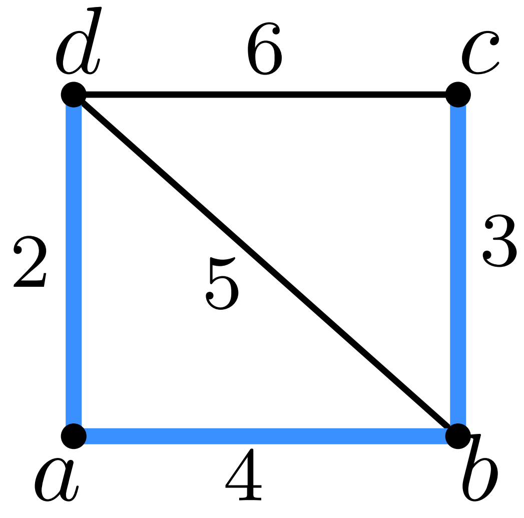
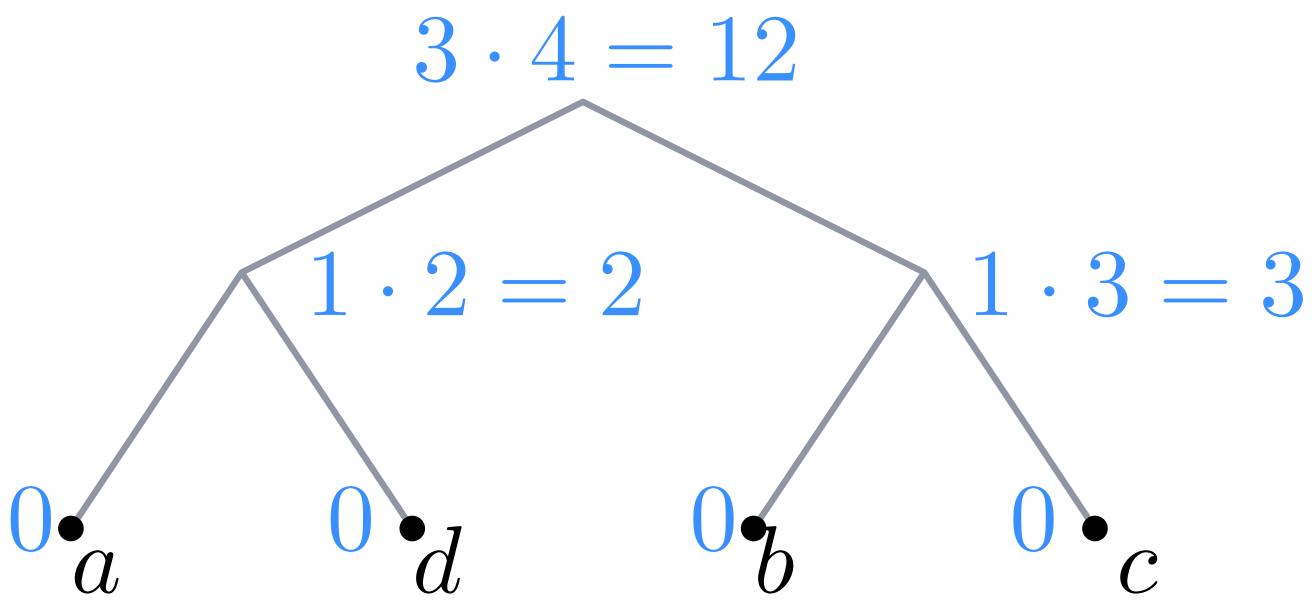
   $d_M(x, y)$
   = weight of shortest path from x to y in G
   $\leq \sum e_i \leq |P_v| \max e_i = \Delta_v$





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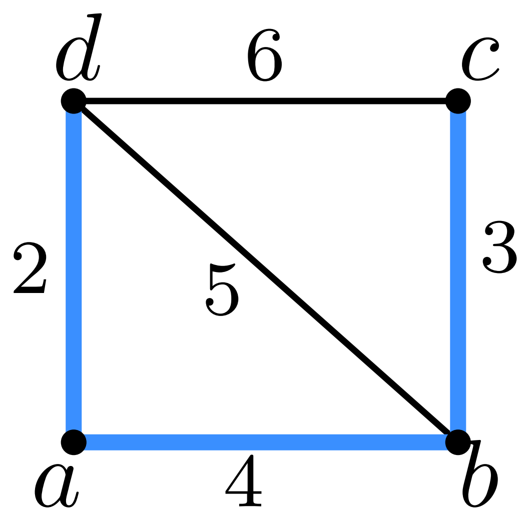
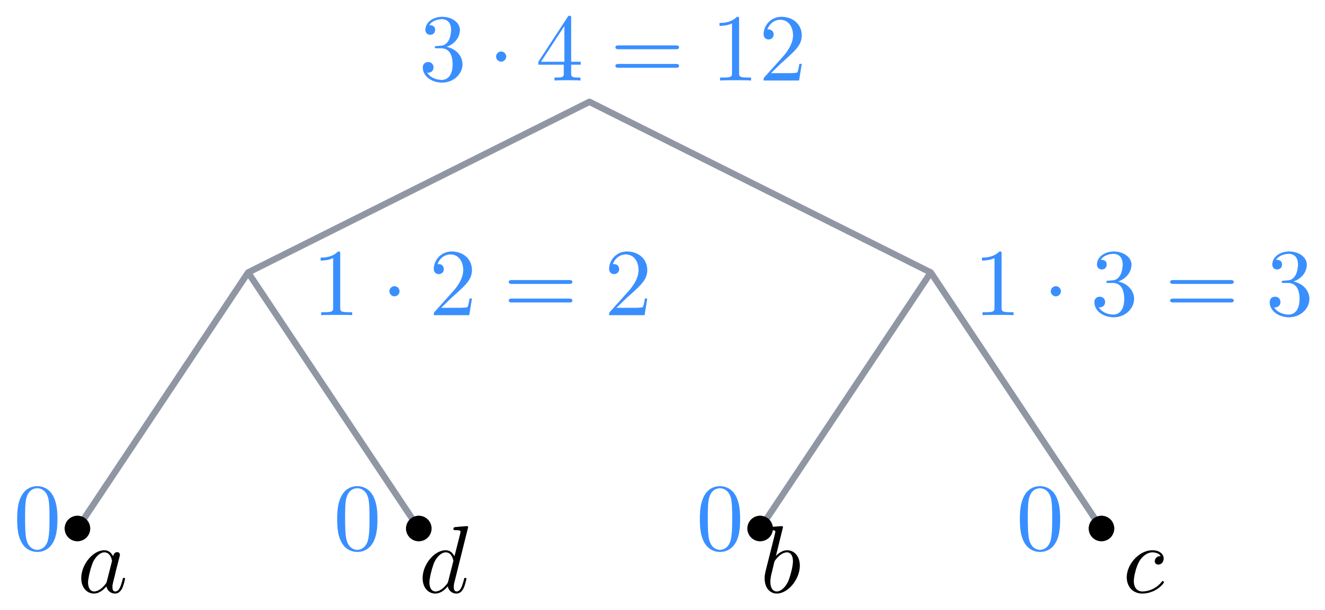
summary:
Given a metric over a set P, we can efficiently construct a hierarchically well-balanced tree that $(n-1)$-approximates the metric.





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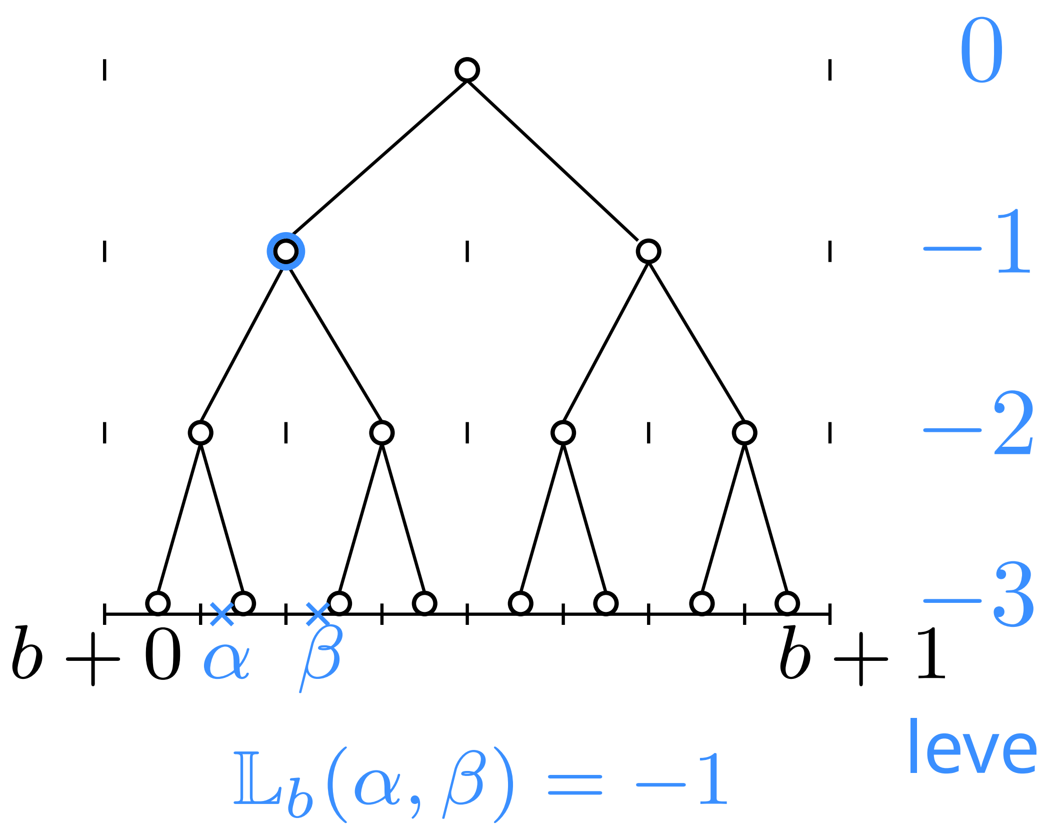
alternative for metric space $\mathbb{R}^d$: shifted quadtree, with Δ_v = diameter of cell



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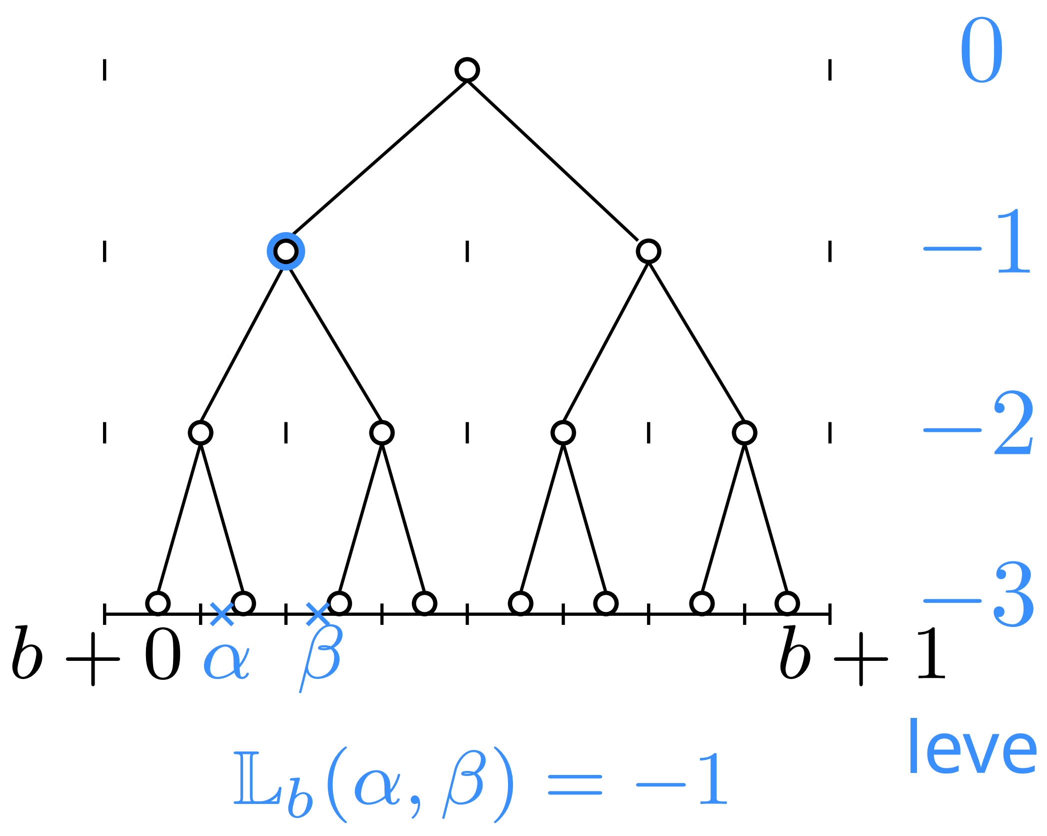

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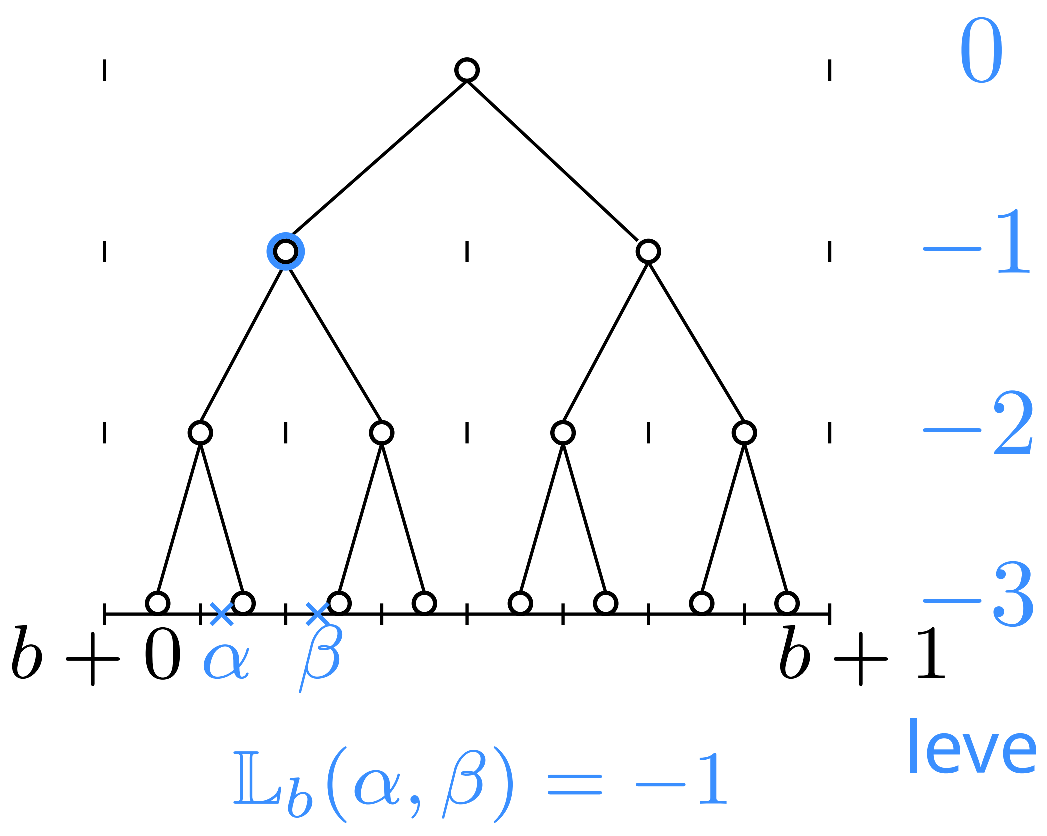

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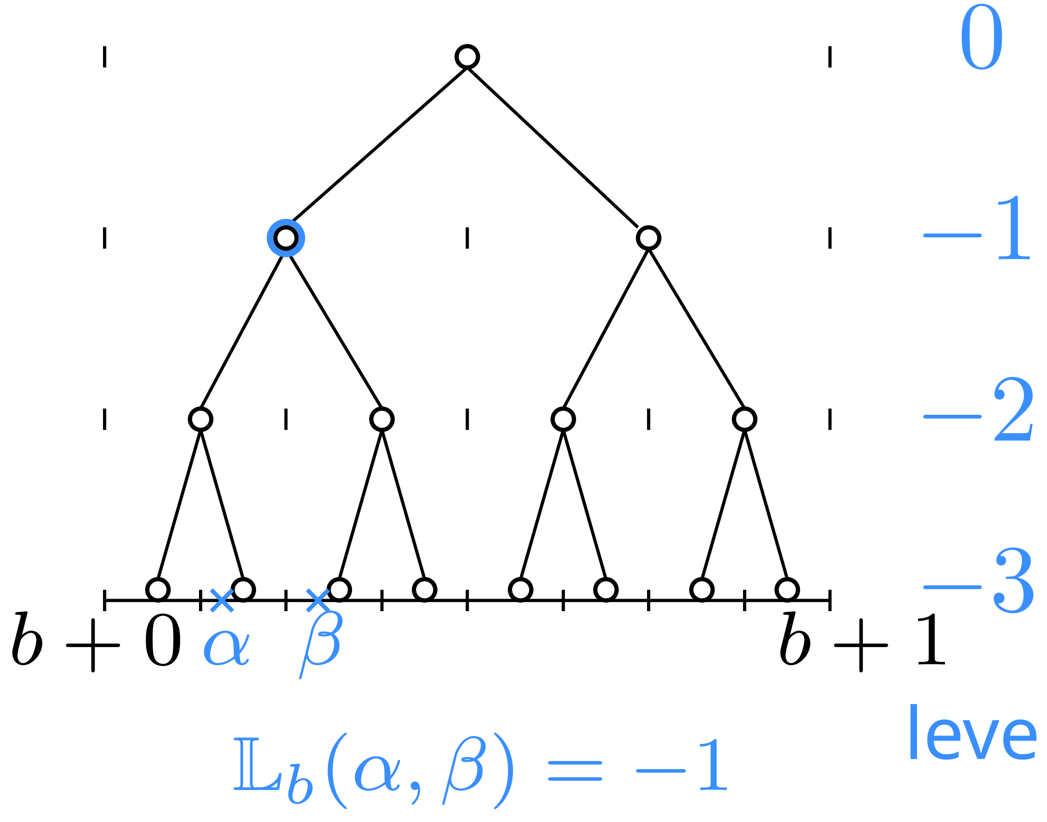
$$= n^{O(1)} d_M(p,q)$$



Overview

Approximating a metric space by a hierarchical well-separated tree (HST)

hierachical well-separated trees

simple $(n-1)$-approximation

fast $n^{O(1)}$-approximation in $\mathbb{R}^d$

ANN via point location among balls

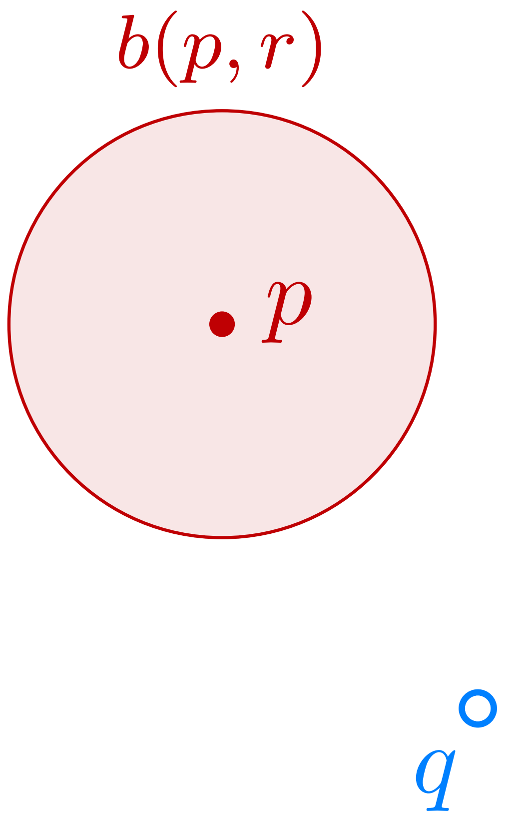
simple construction

handling a range of radii

ANN data structure based on HST

Point Location among balls

$b(p, r) = \{q \mid d(p, q) \leq r\}$ denotes a ball around point p with radius r



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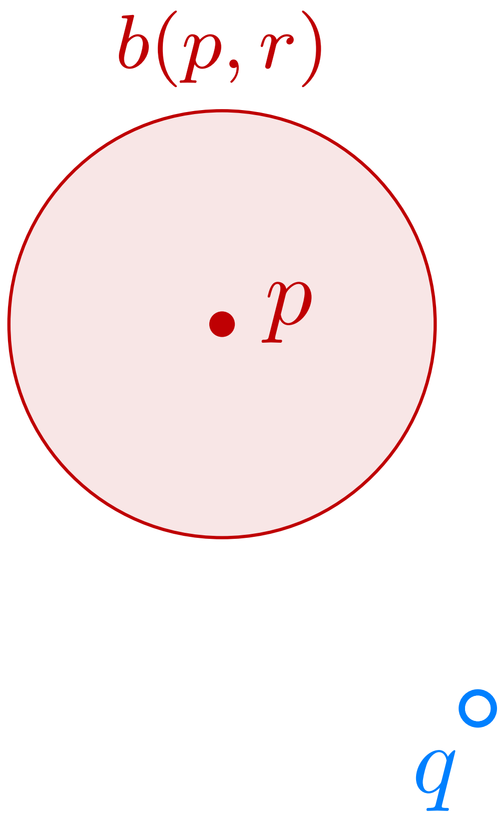
point set P in a *metric space* $\mathcal{M}$



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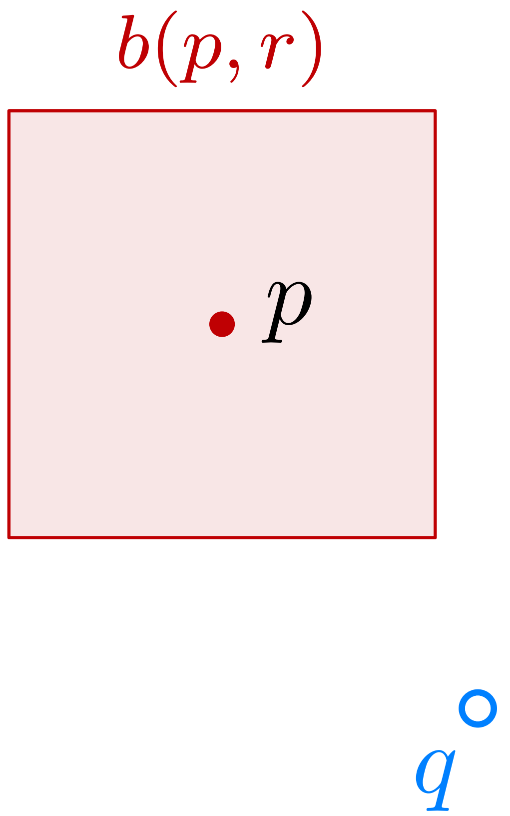

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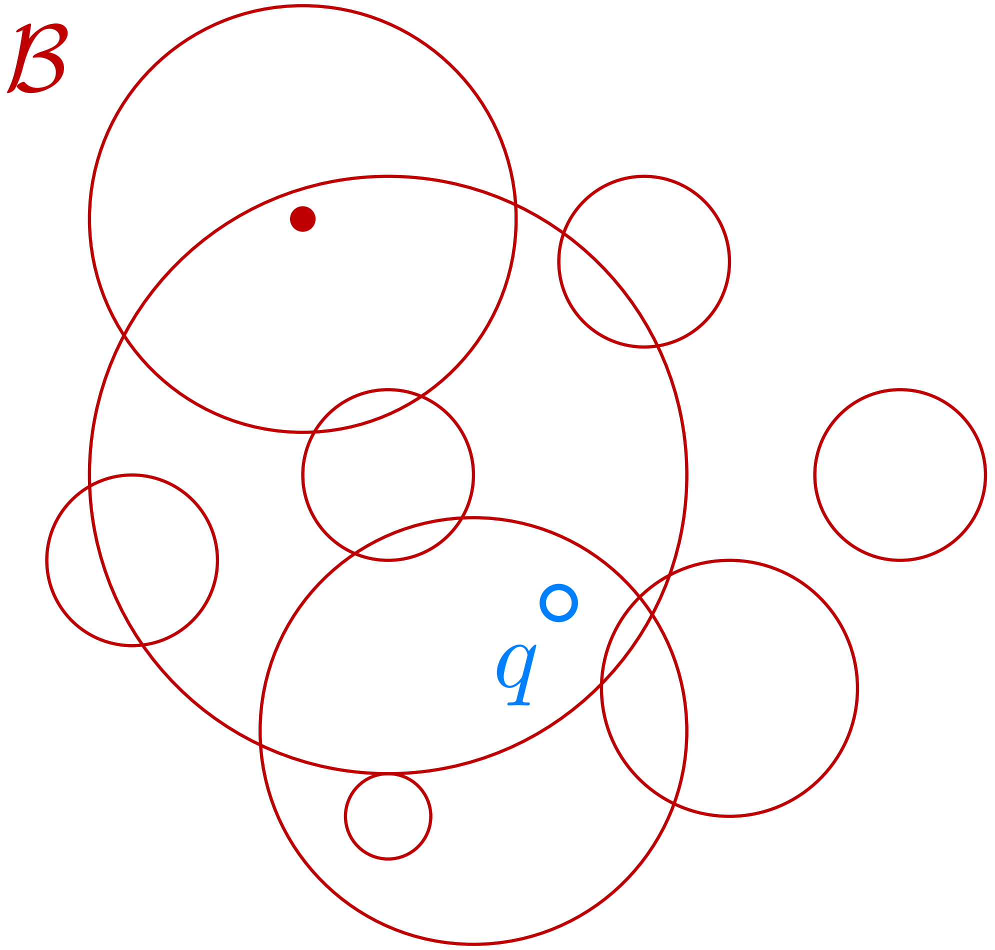
For a collection of balls $\mathcal{B}$, $\odot_{\mathcal{B}}(q)$ denotes the smallest ball containing q, or target ball for q



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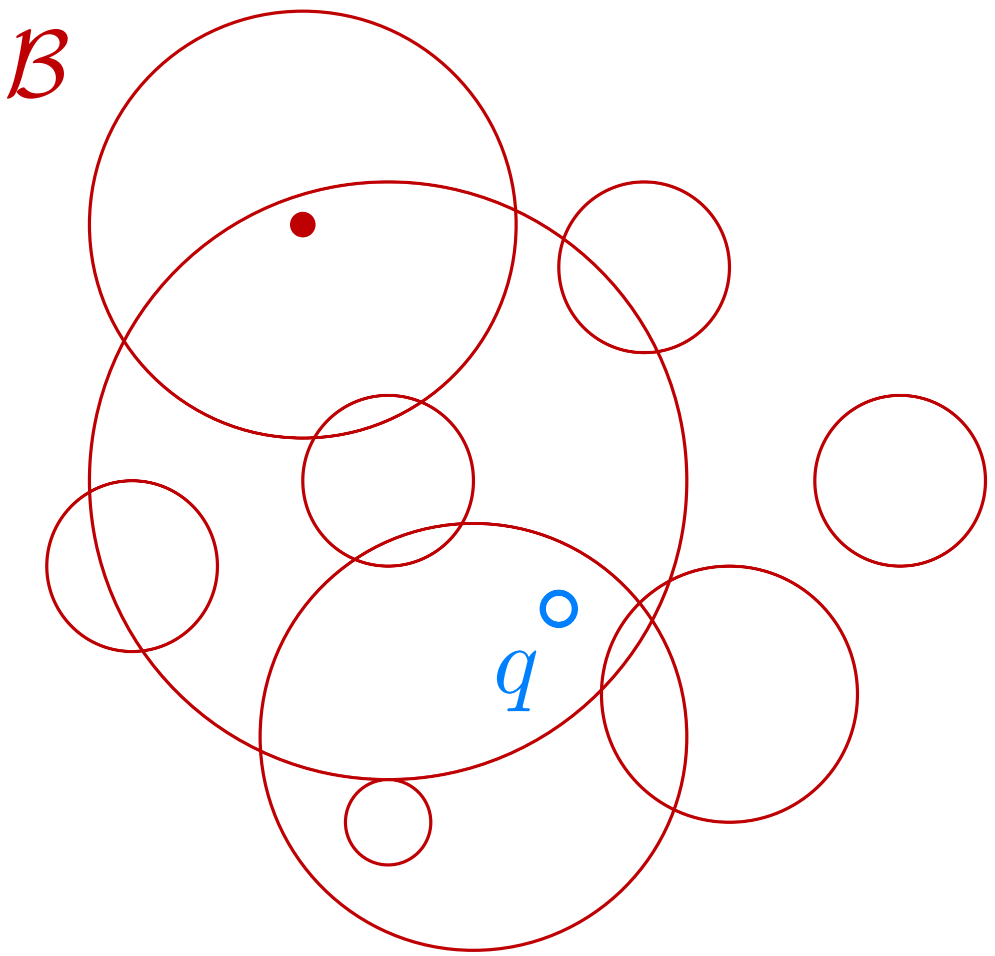

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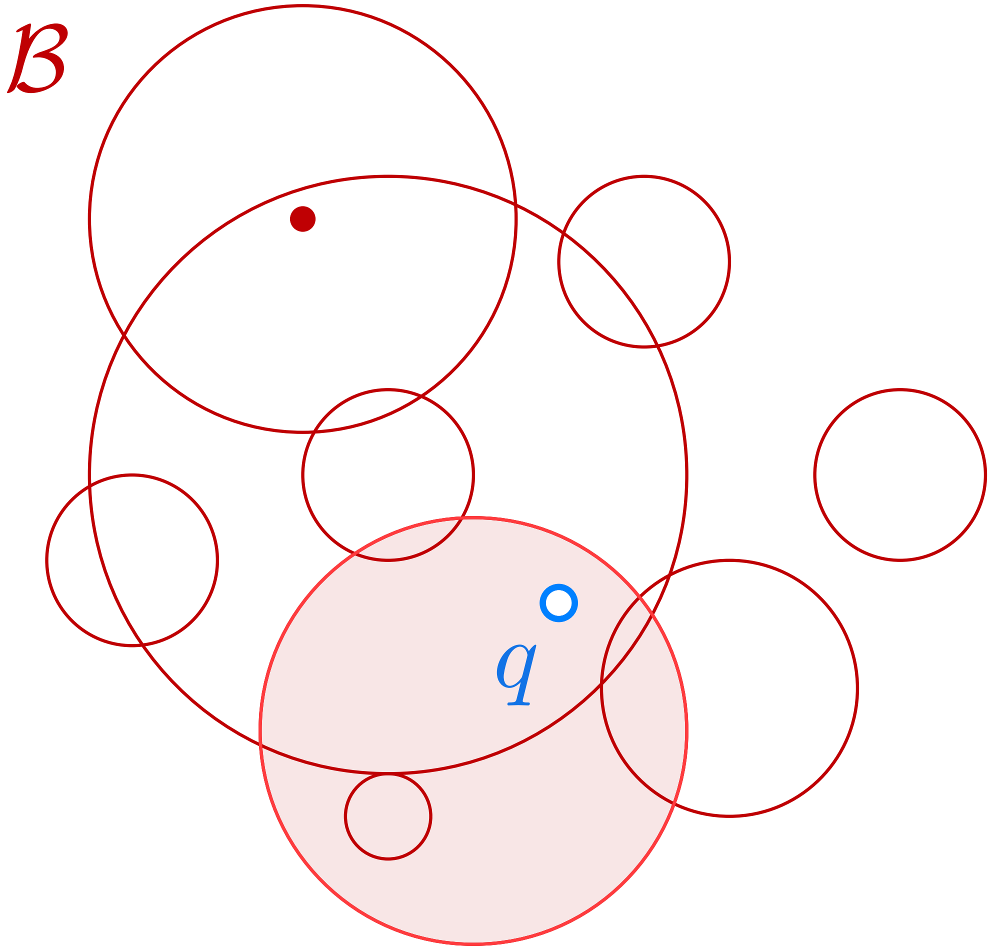
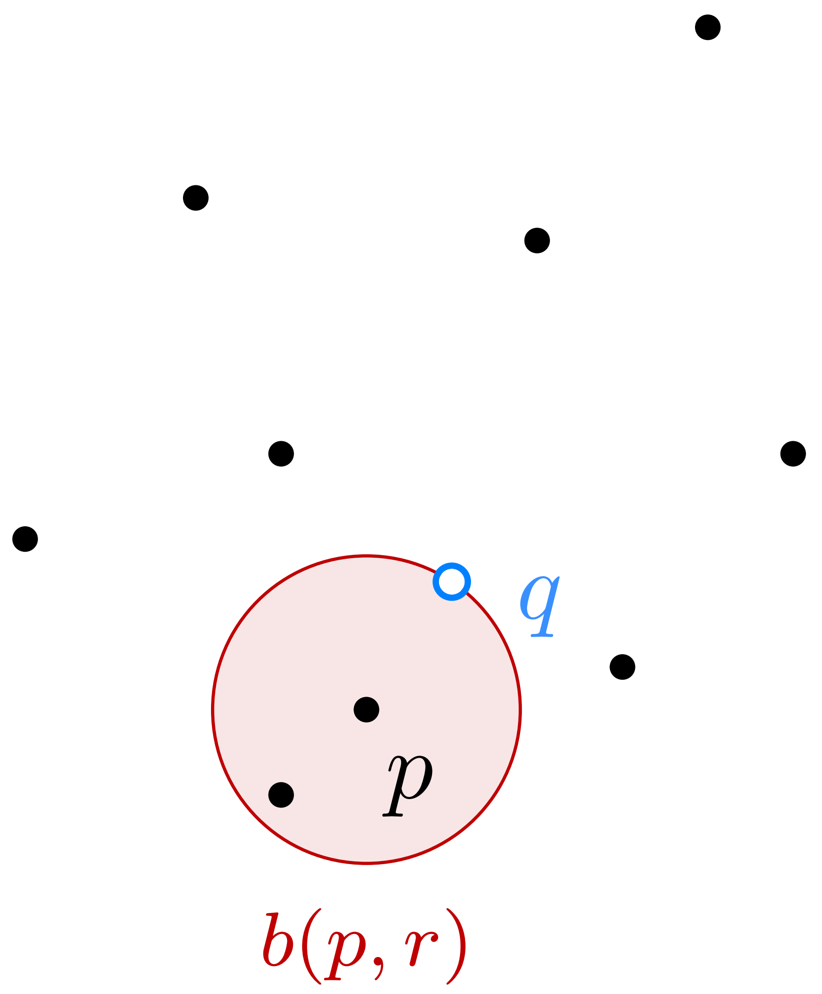


Relation $(1+\varepsilon)$-**ANN** and **PLEB**

Gaining some intuition:



Relation $(1+\varepsilon)$-ANN and PLEB

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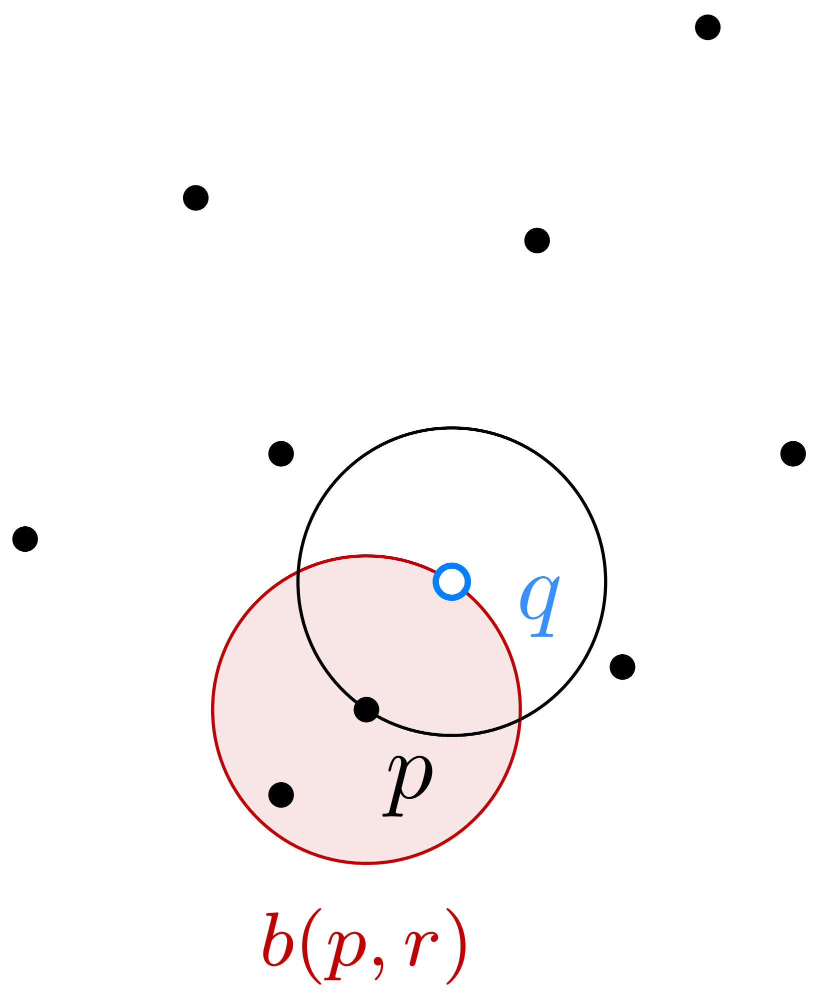
$$r = d(q,p) \geq d(q,P)$$



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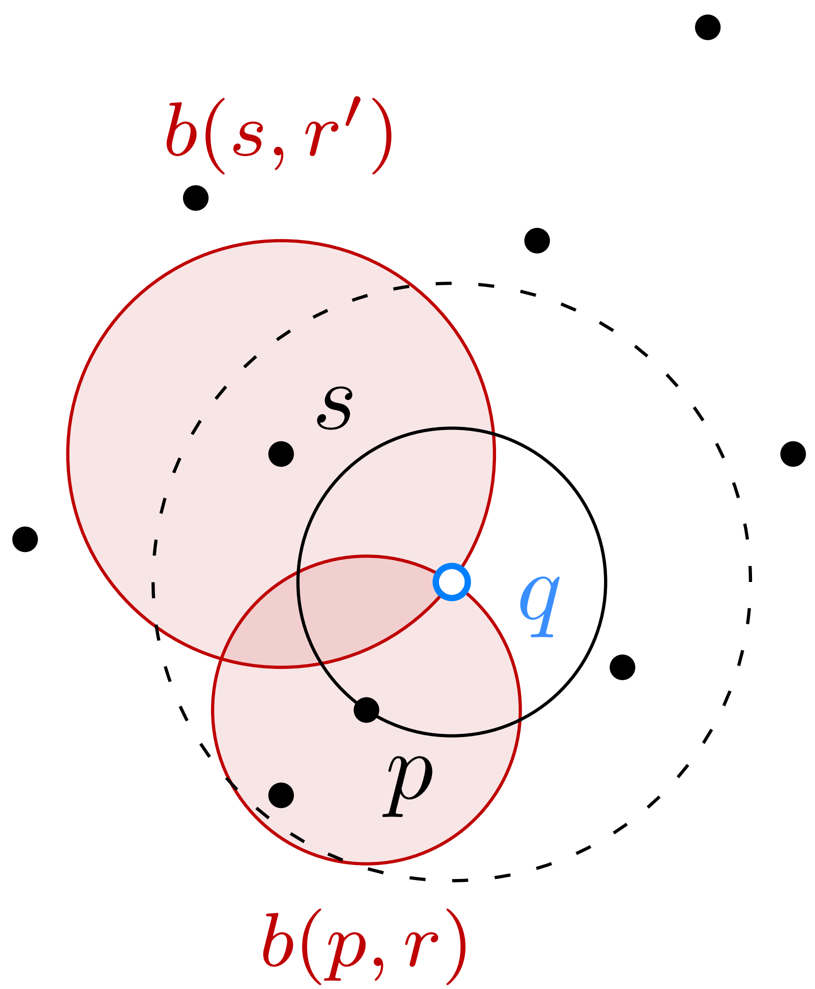
$$r = d(q,p) \geq d(q,P)$$

$$r' = d(q,s) \leq (1+\varepsilon)d(q,P)$$



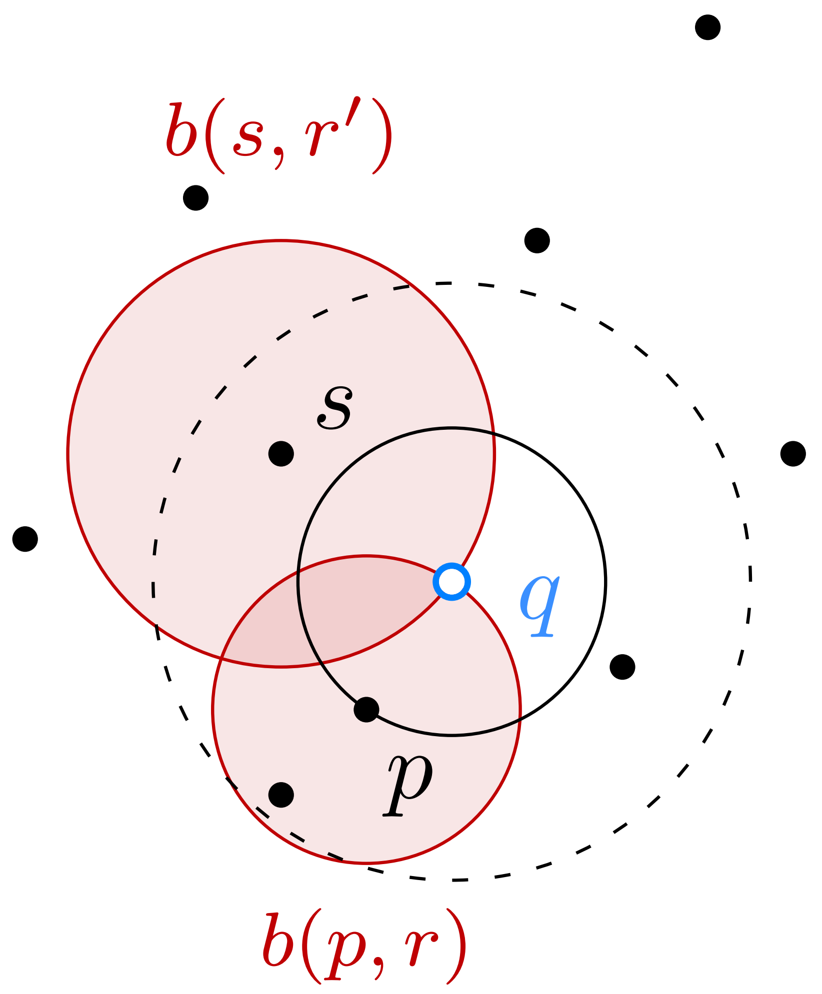
Relation $(1+\varepsilon)$-**ANN** and **PLEB**

Gaining some intuition:

$r = d(q,p) \geq d(q,P)$

$r' = d(q,s) \leq (1+\varepsilon)d(q,P)$

since $r \leq r'$, we have $d(q,p) \leq (1+\varepsilon)d(q,P)$



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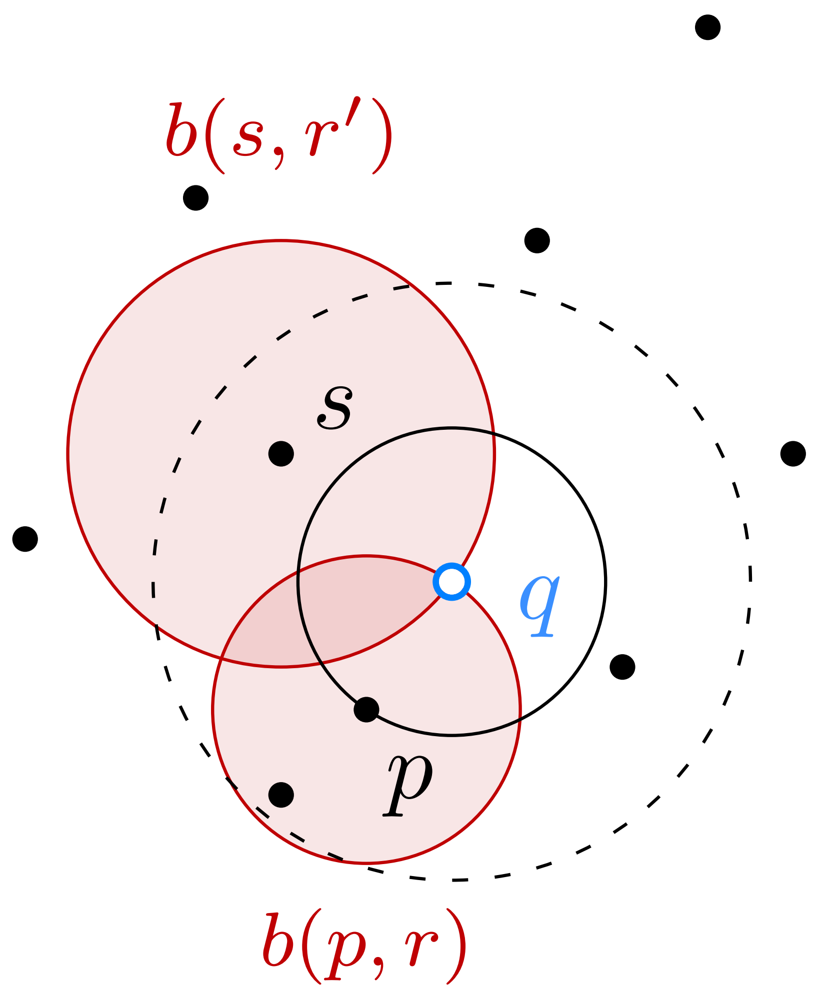

Relation $(1+\varepsilon)$-**ANN** and **PLEB**

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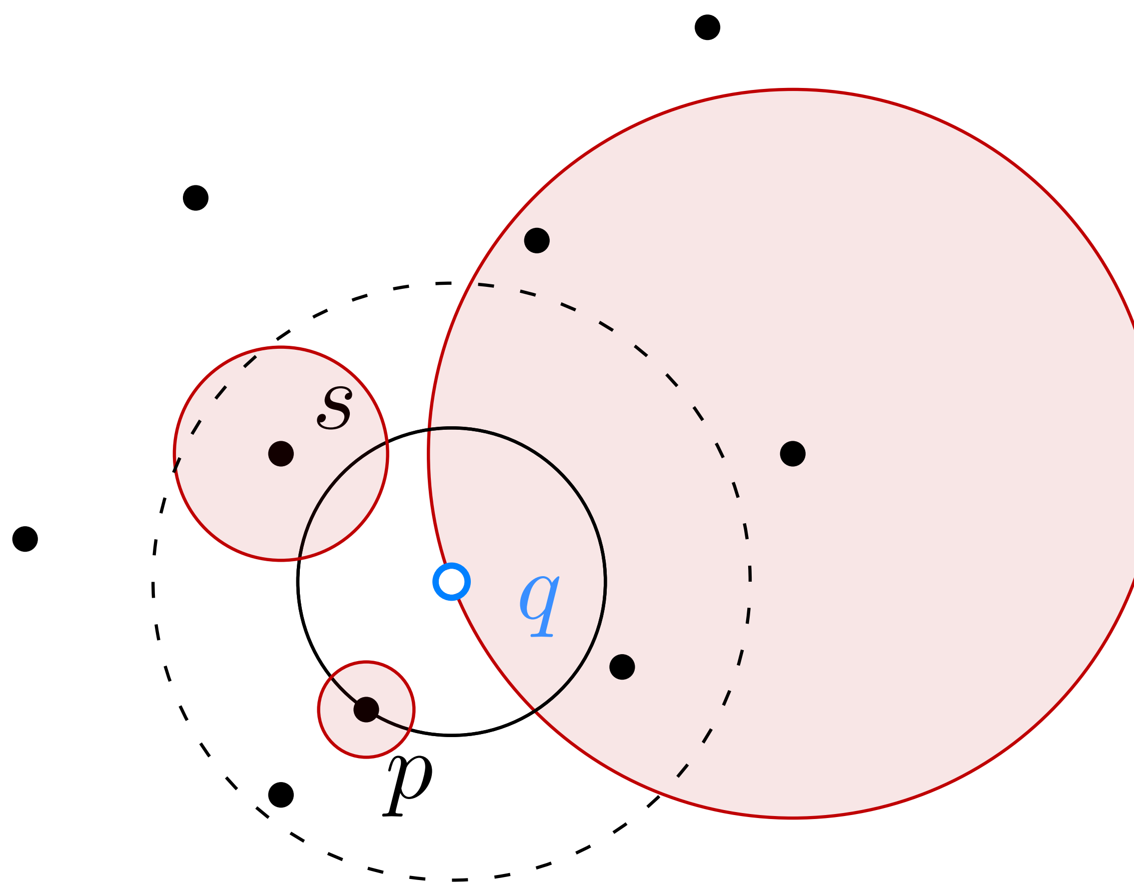
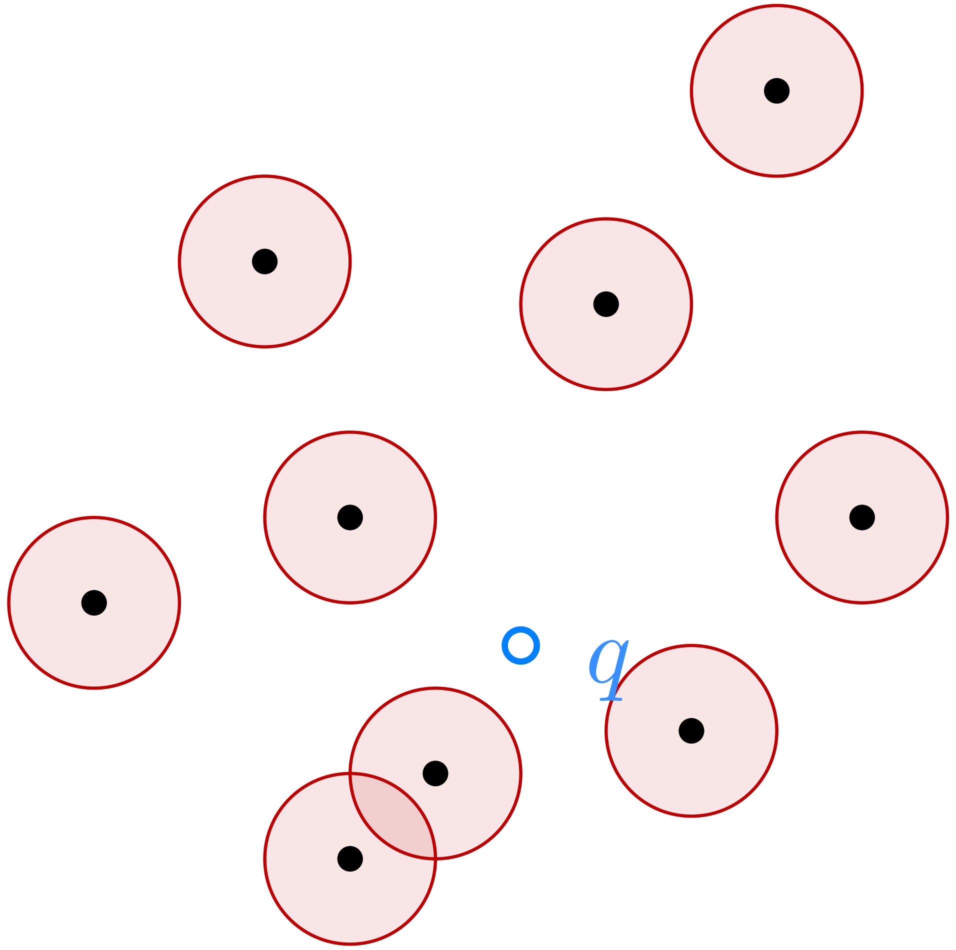


Reduction

Reduction from $(1+\varepsilon)$**-ANN** to **Point location among balls**

define $\mathcal{U}(P,r) = \bigcup_{p \in P} b(p,r)$

union of balls of radius r



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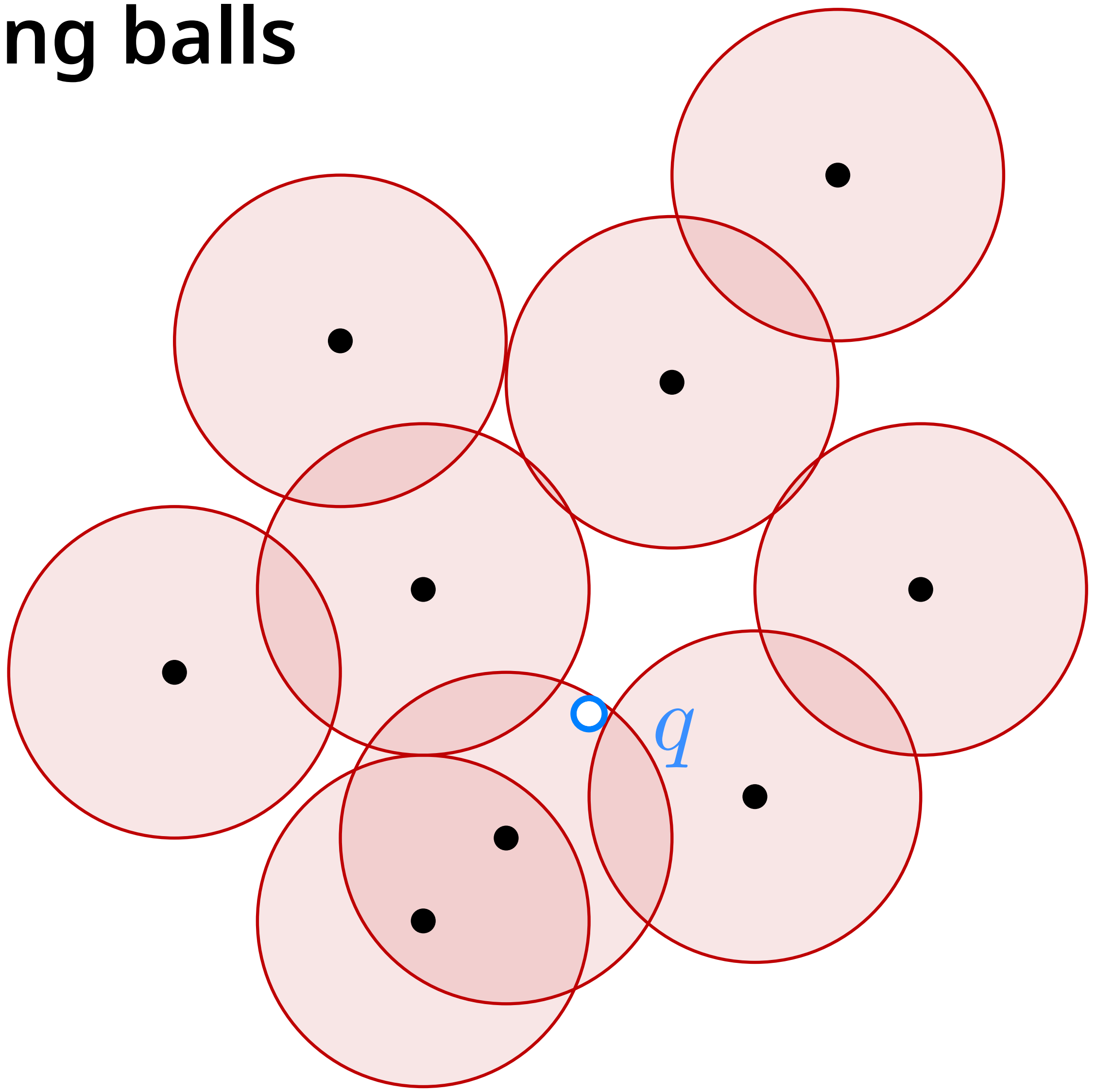

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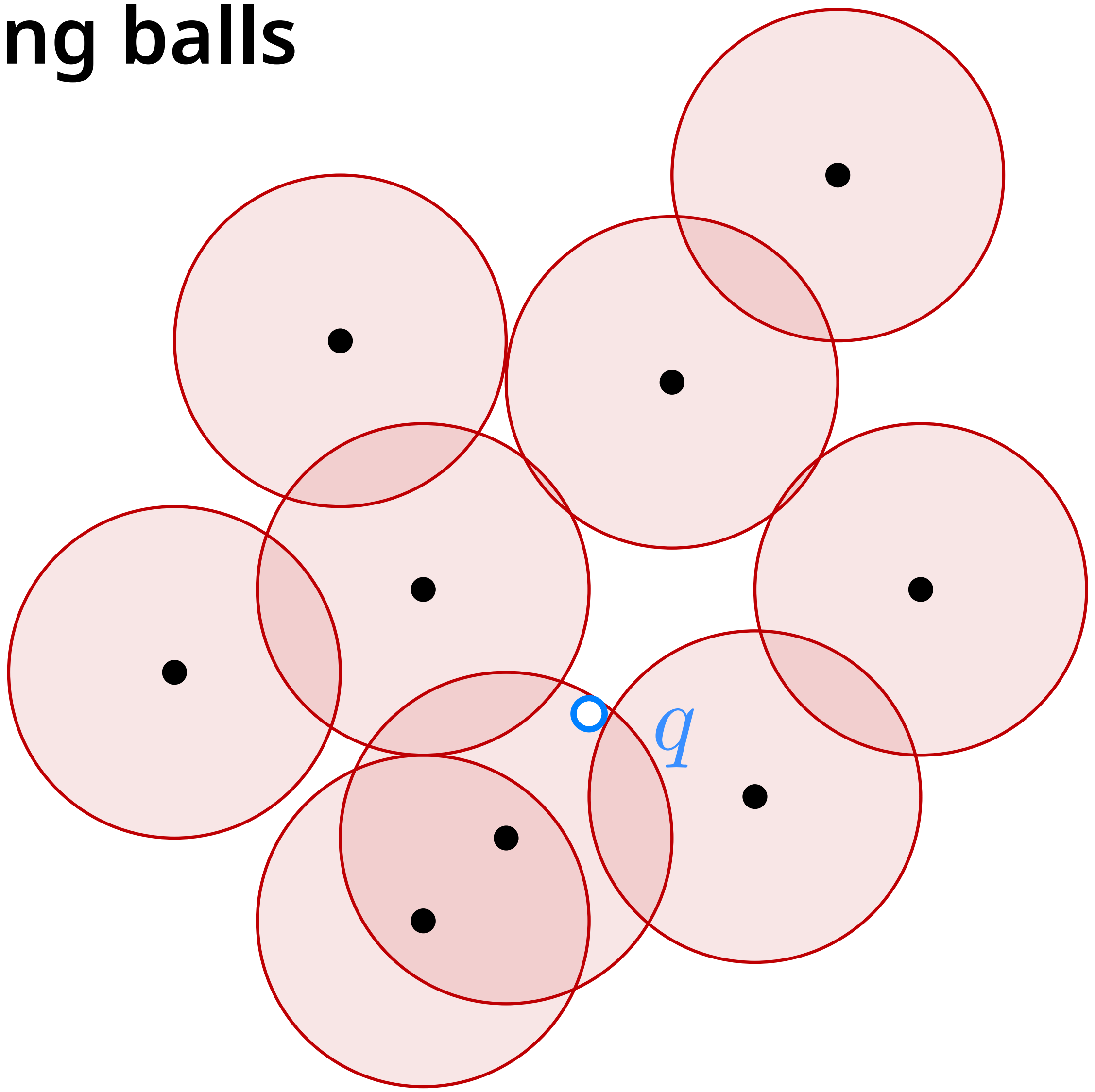

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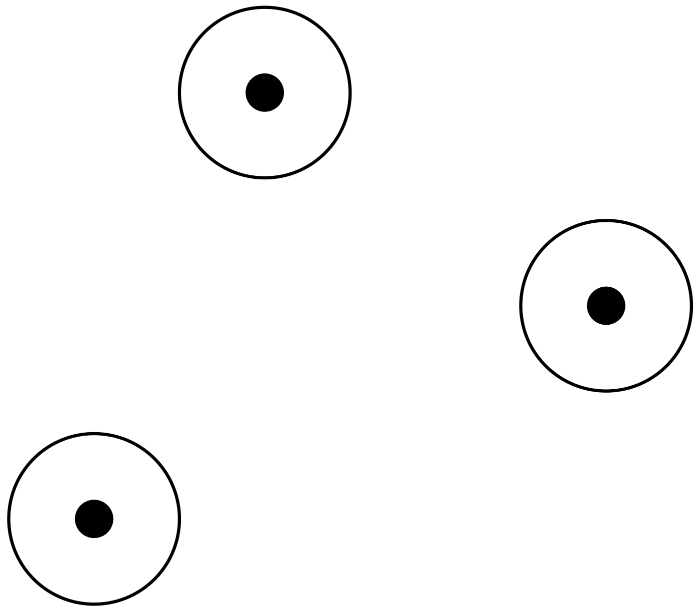

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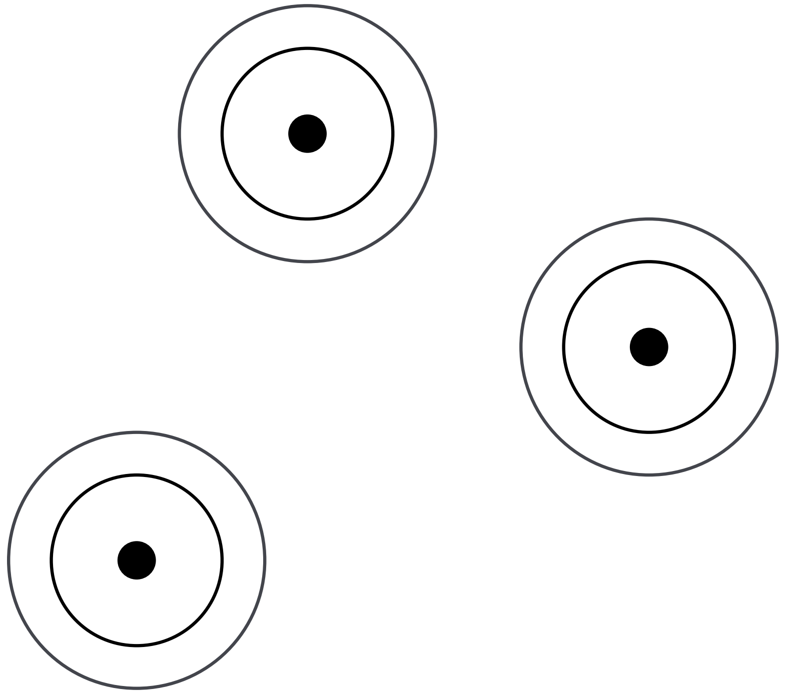

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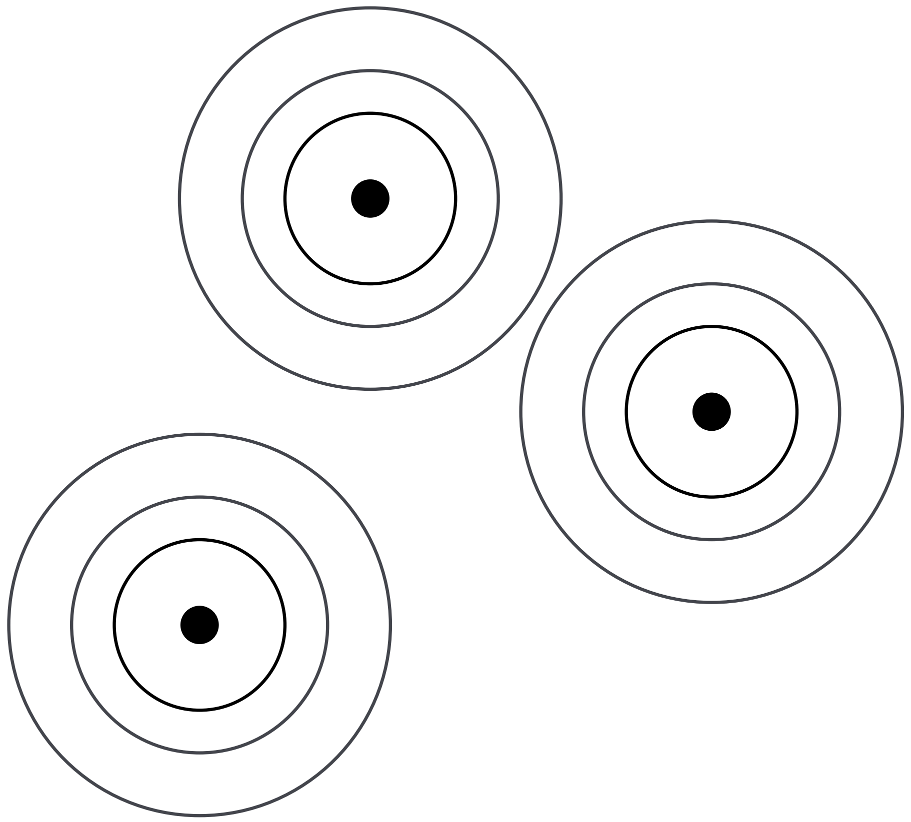

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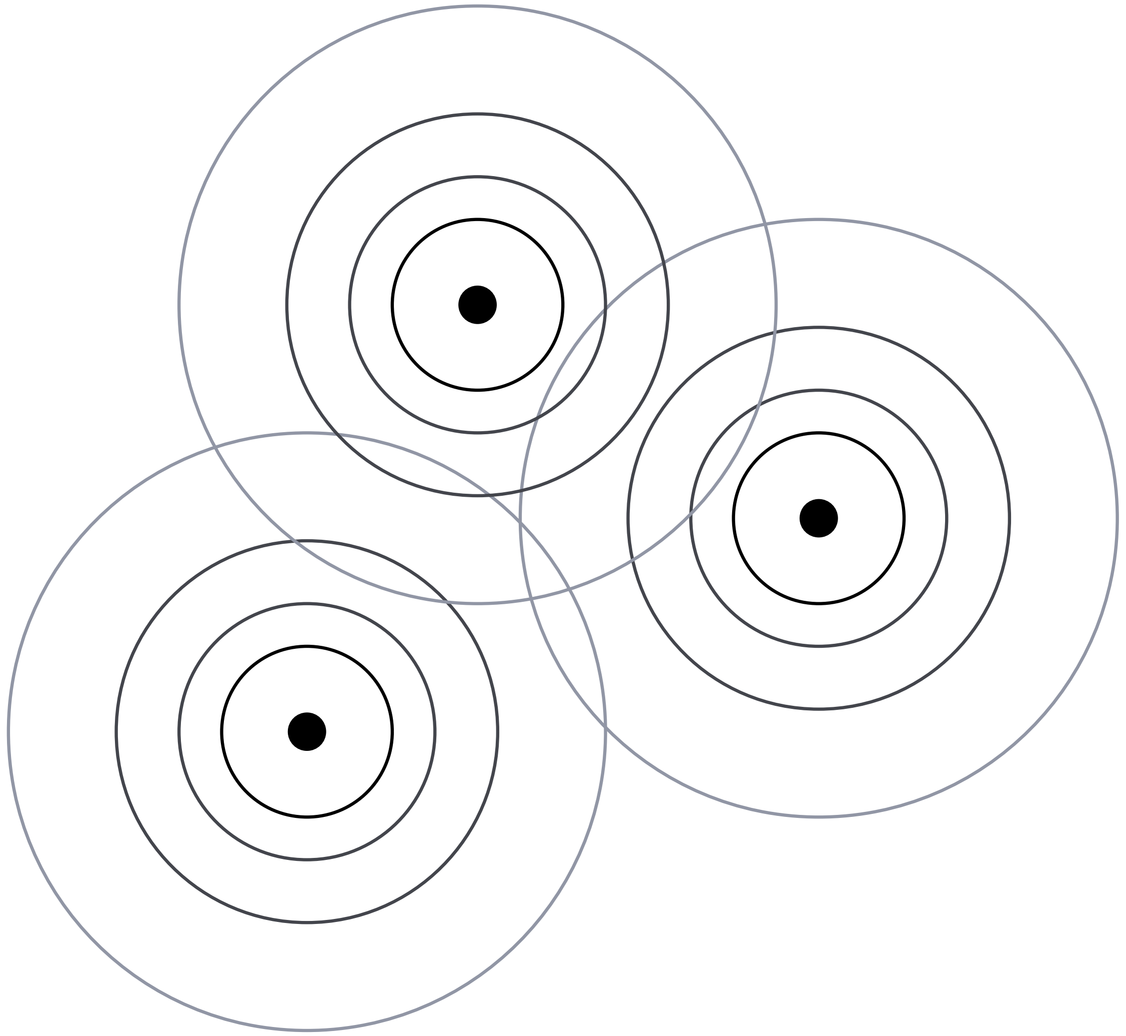

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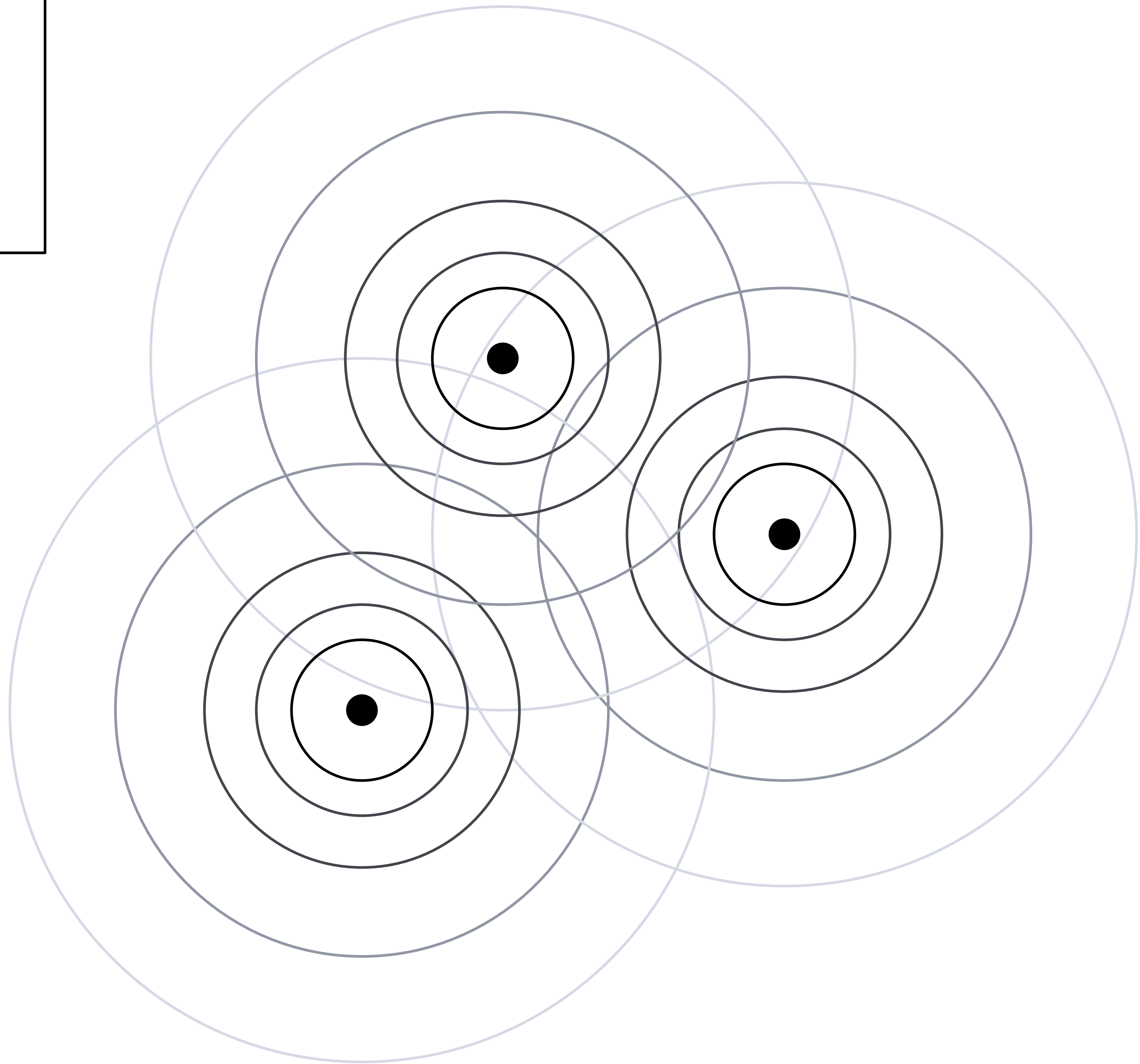

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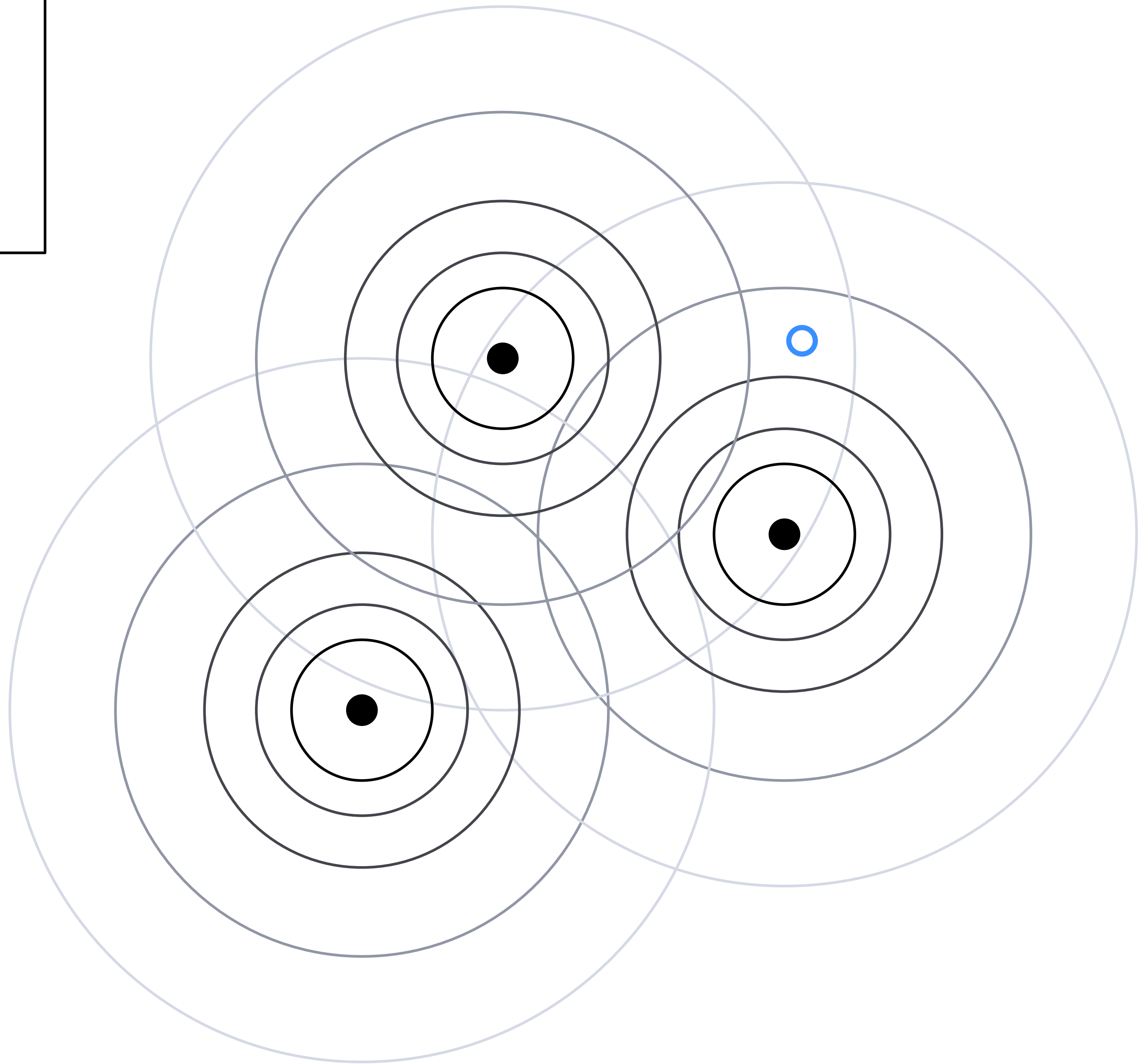

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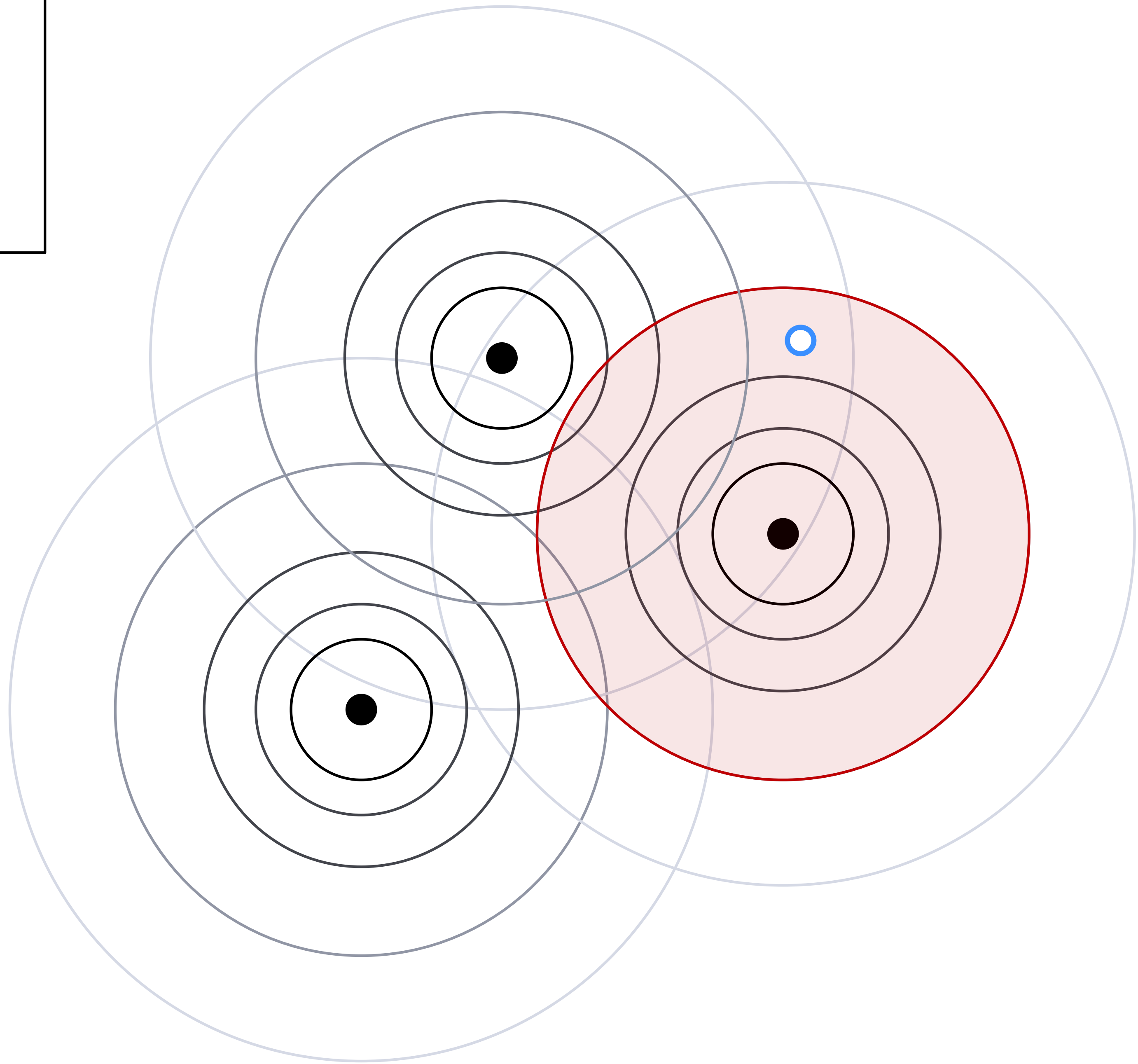

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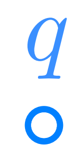

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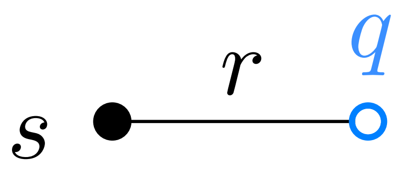

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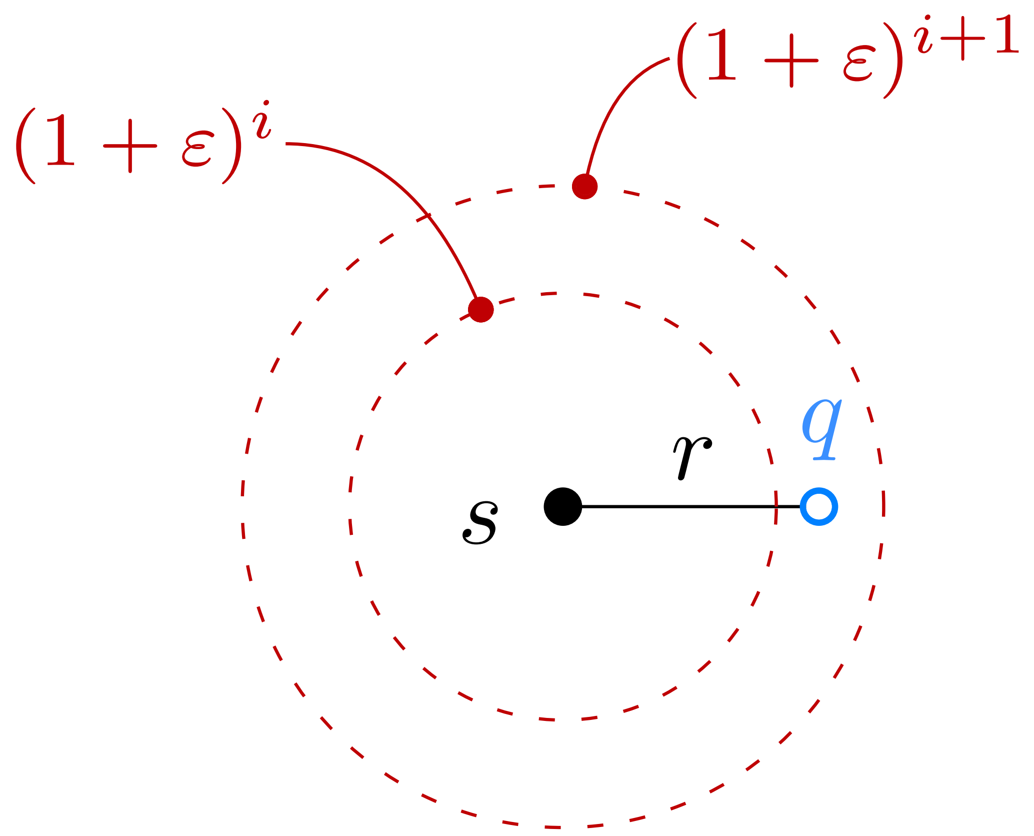

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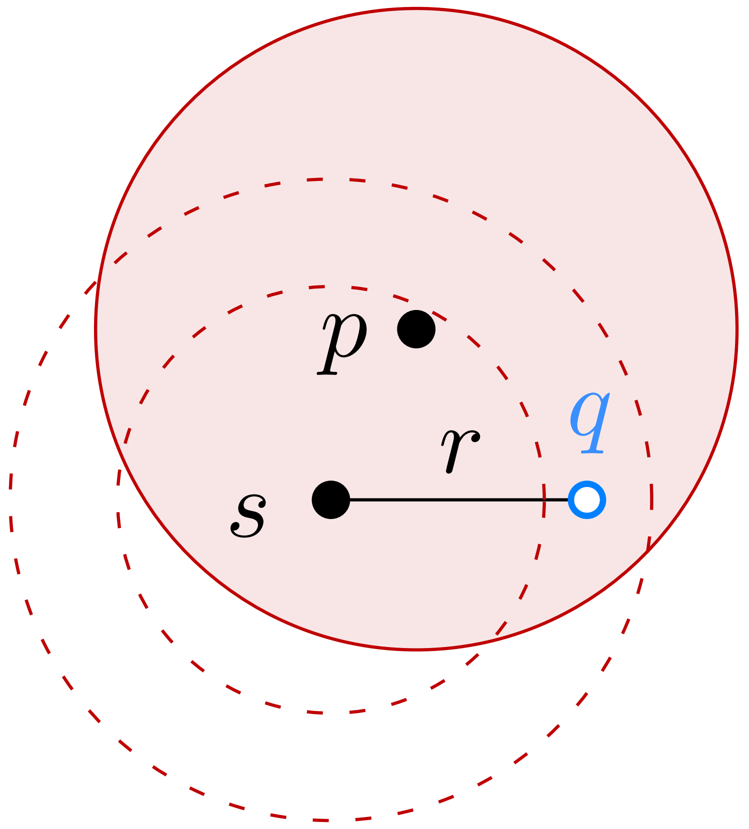

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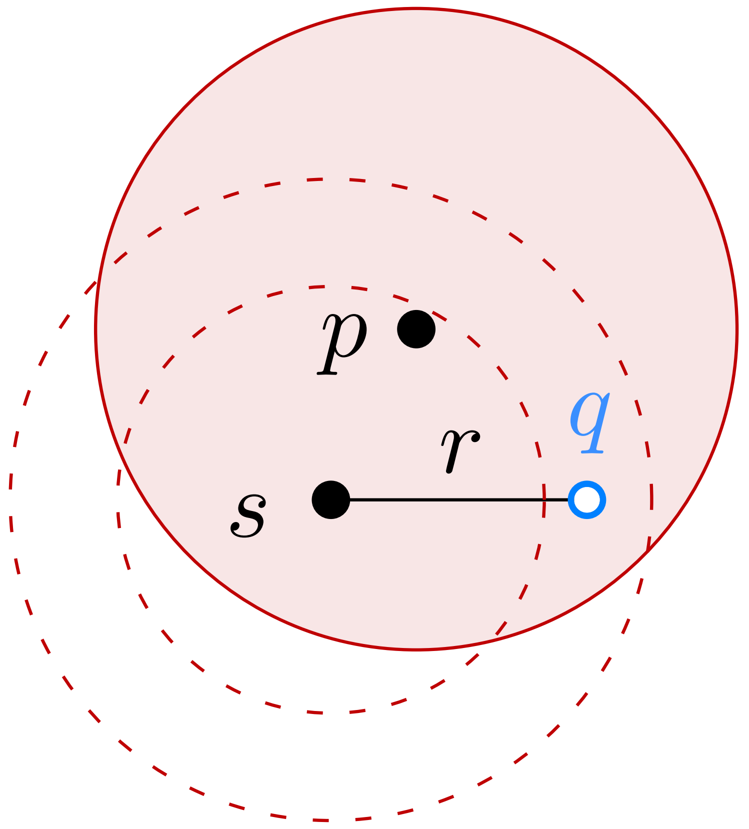

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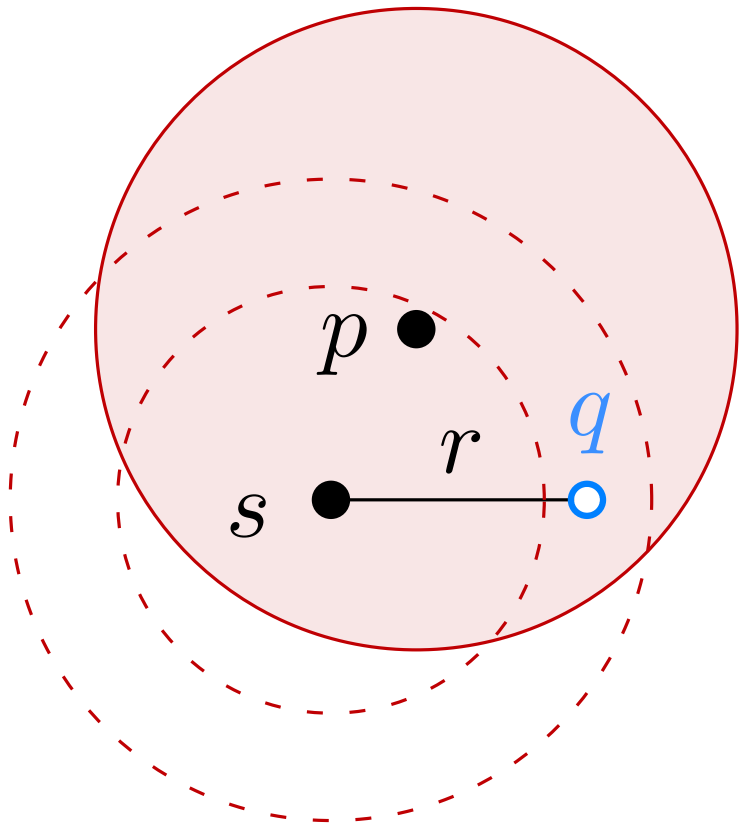

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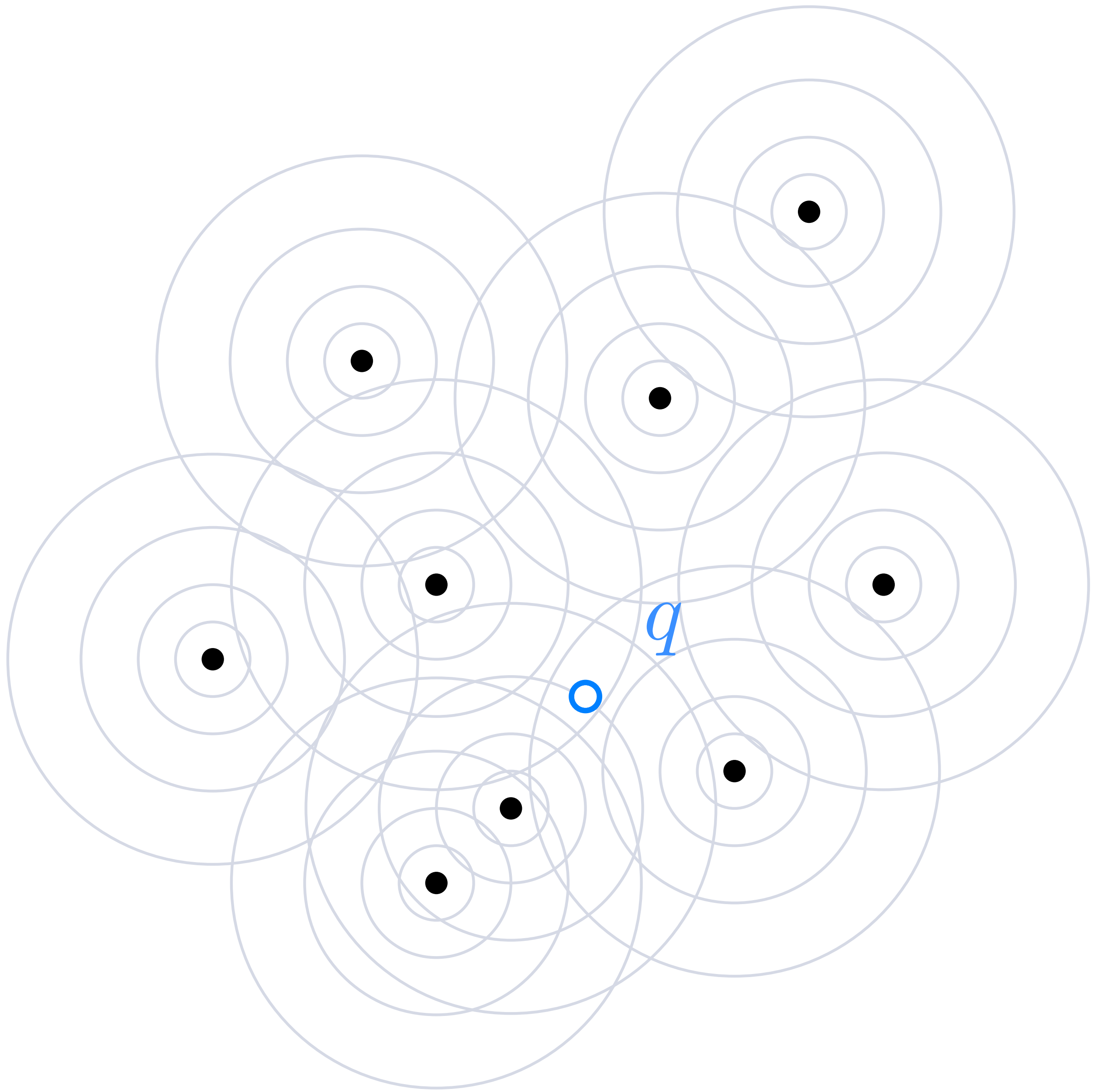

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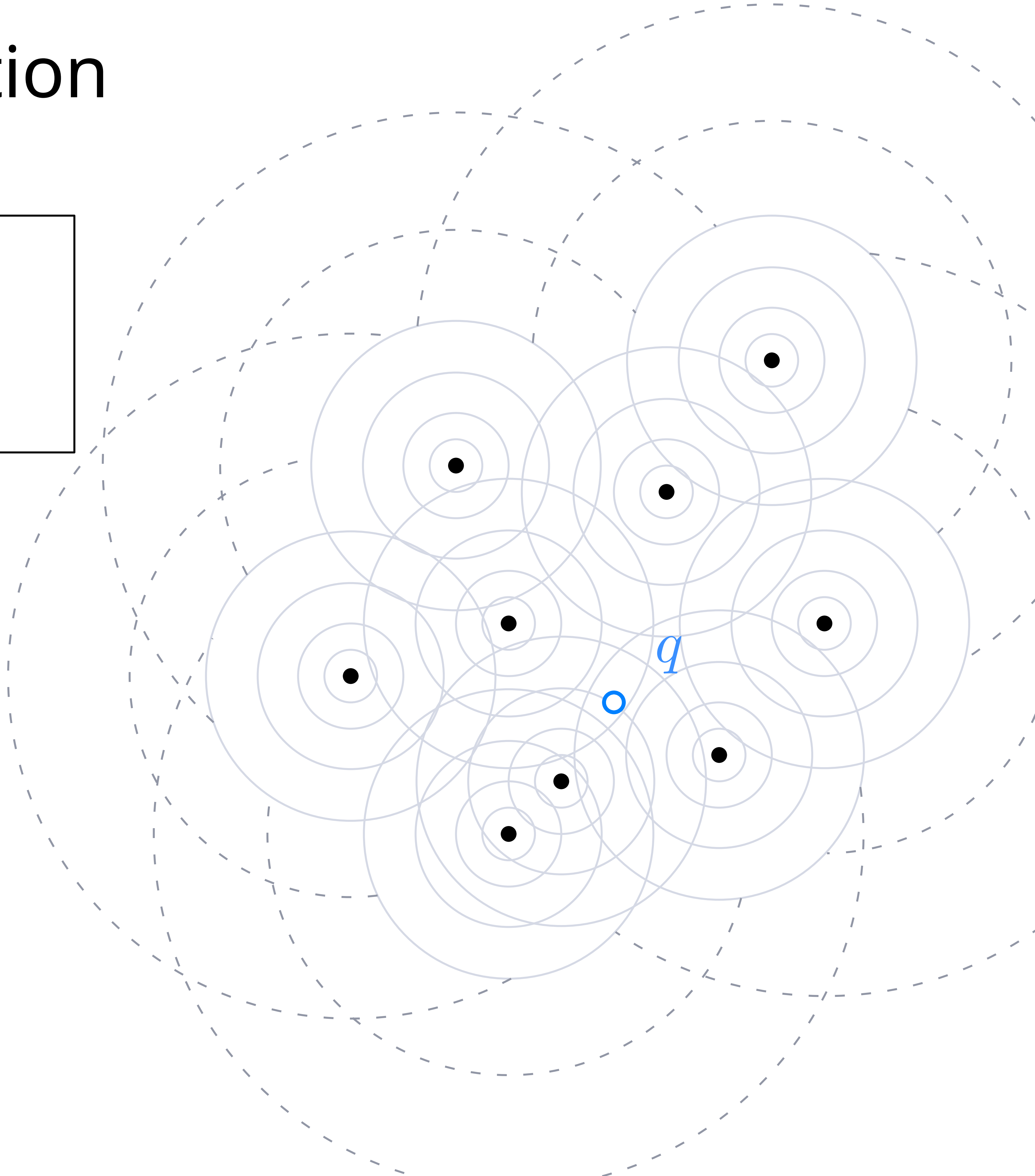

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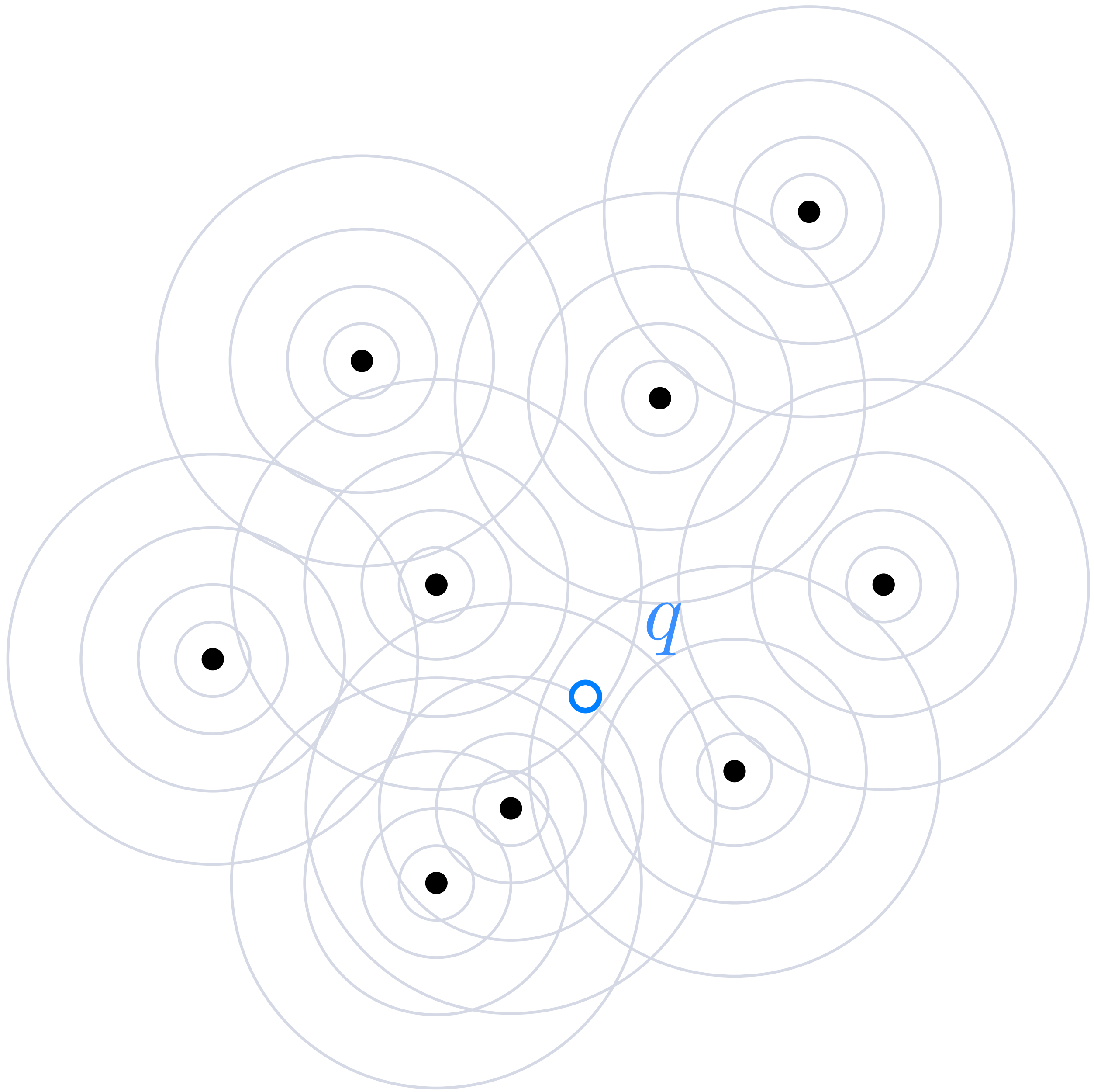

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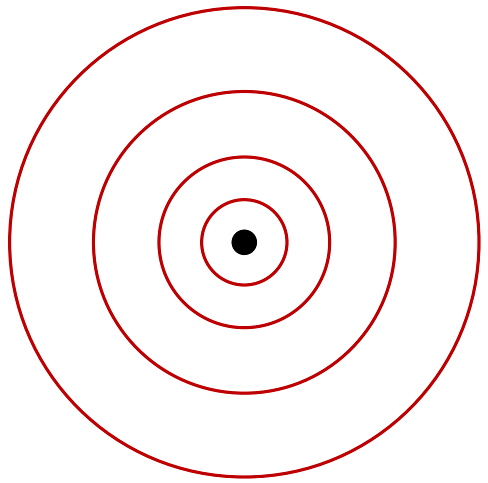

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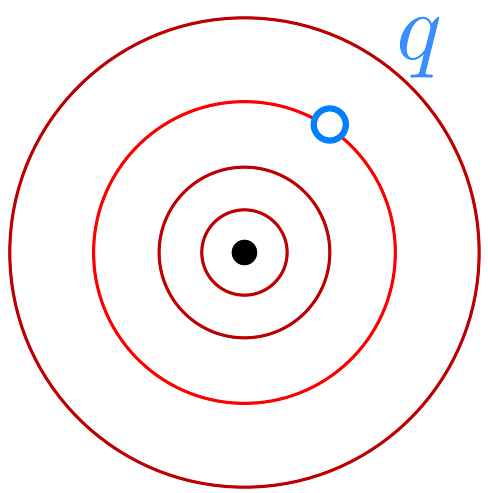

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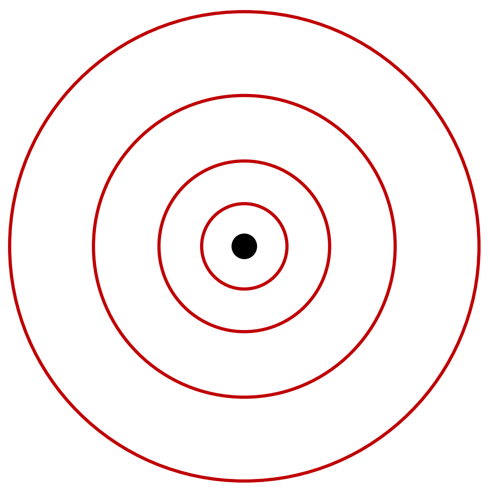

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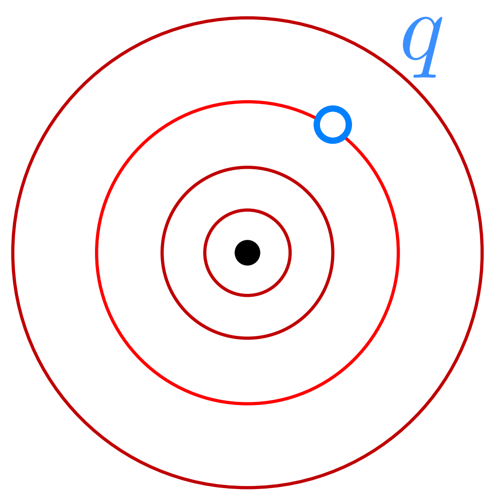

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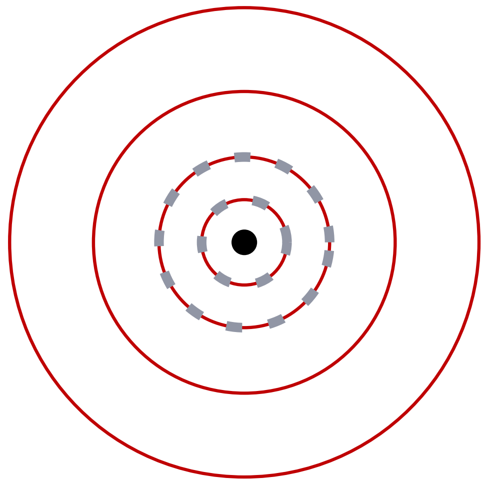

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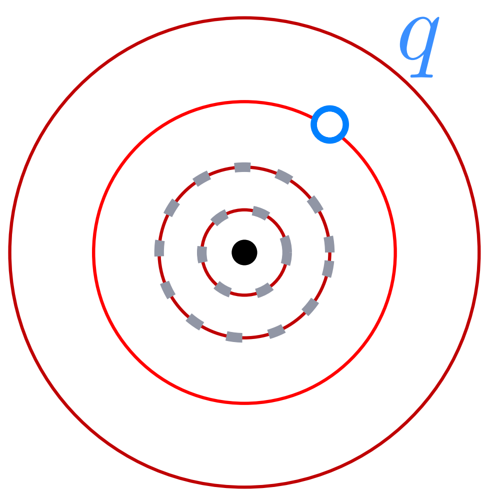

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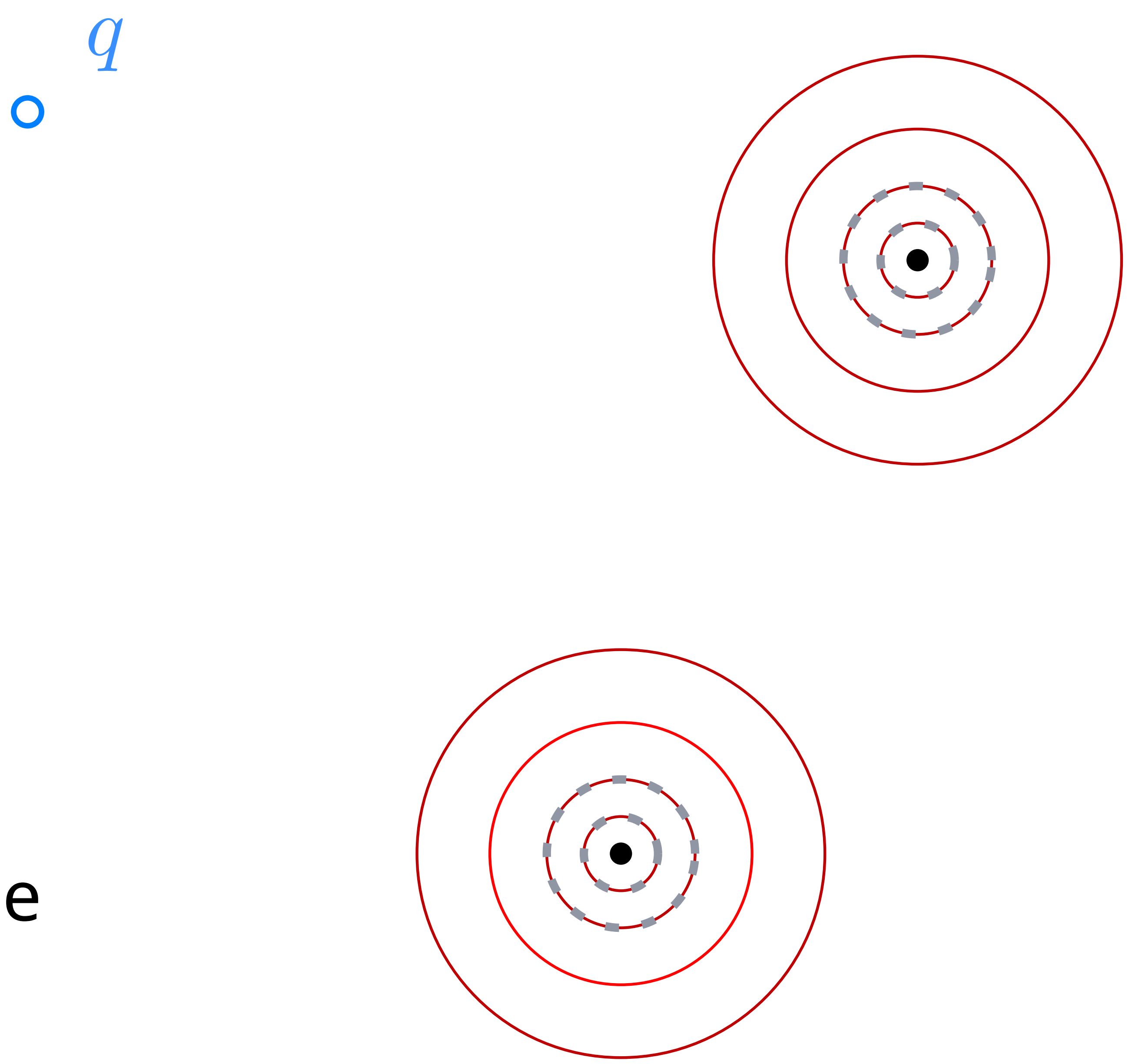

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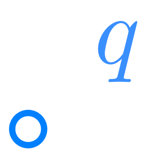

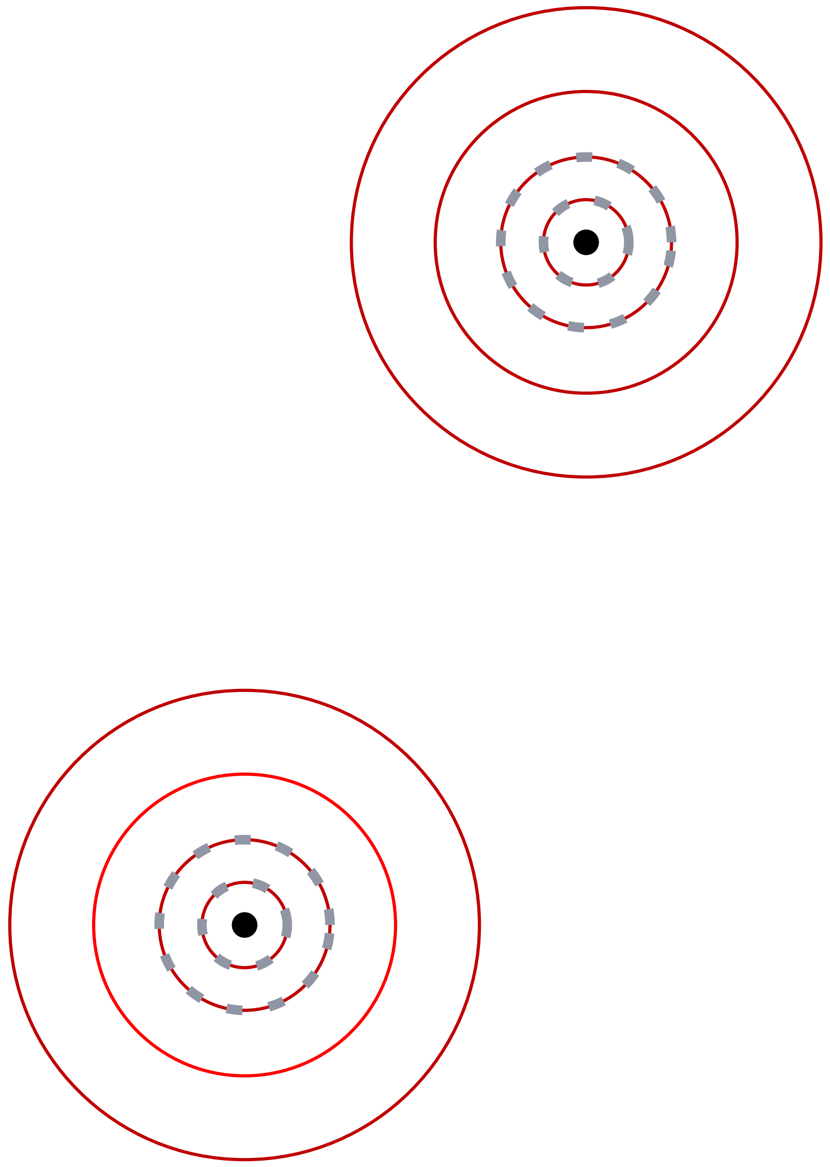

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(1) We only need range of radii:
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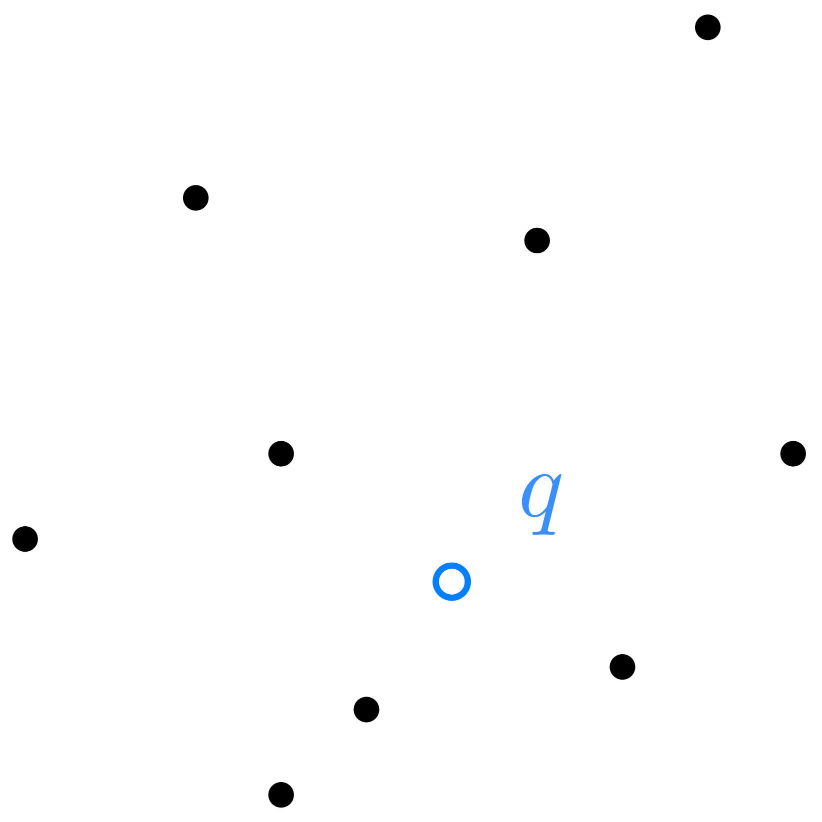
(2) need to avoid range dependent on pairs, otherwise $\Theta(n^2)$ disks

Handling a range of radii

Near neighbor data structure $\mathcal{D}(P, r)$



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Decides given q, if

$d(q, P) \leq r$, or

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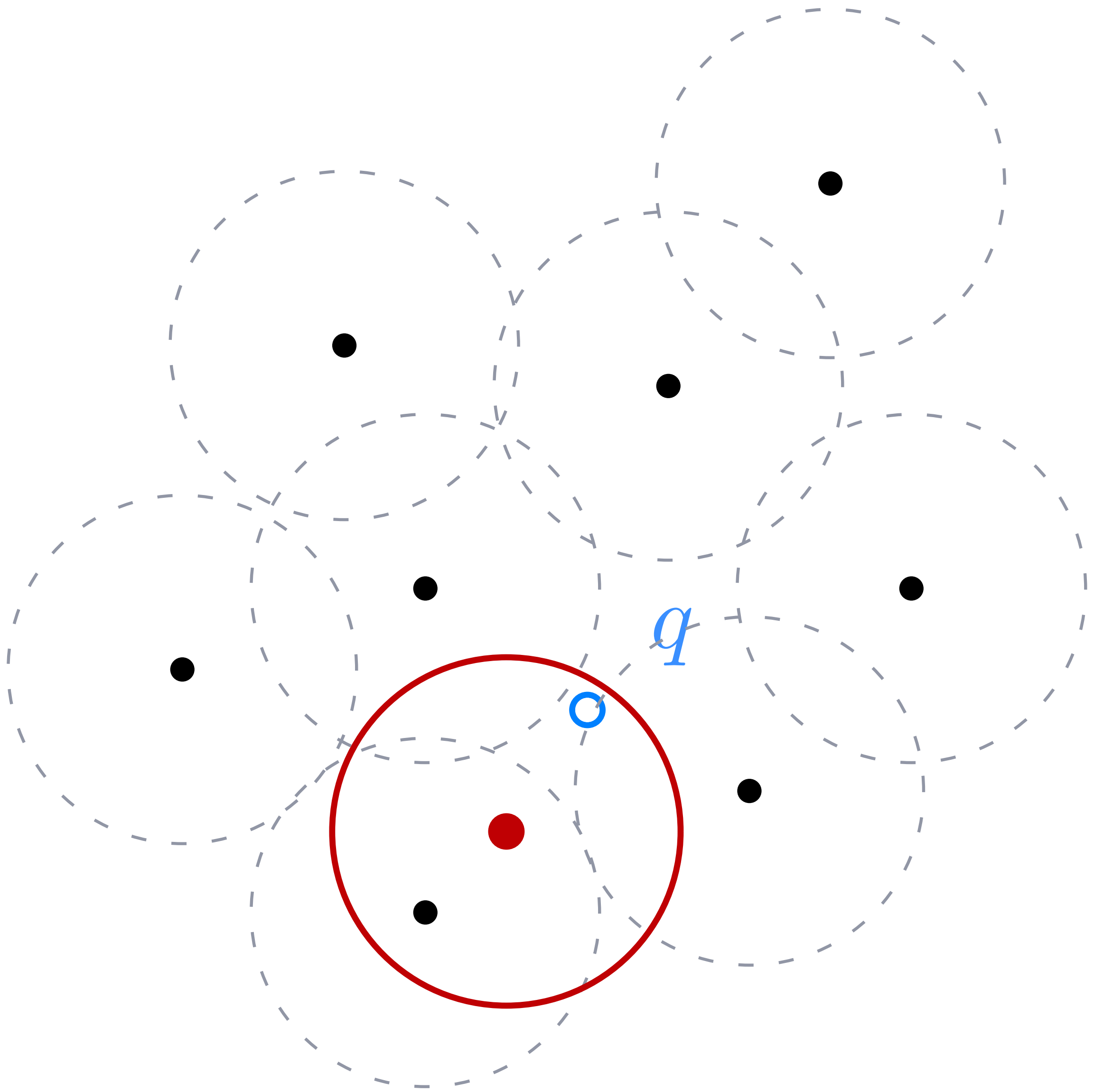
If $d(q, P) \leq r$ it returns a point p s.t. $d(q, p) \leq r$



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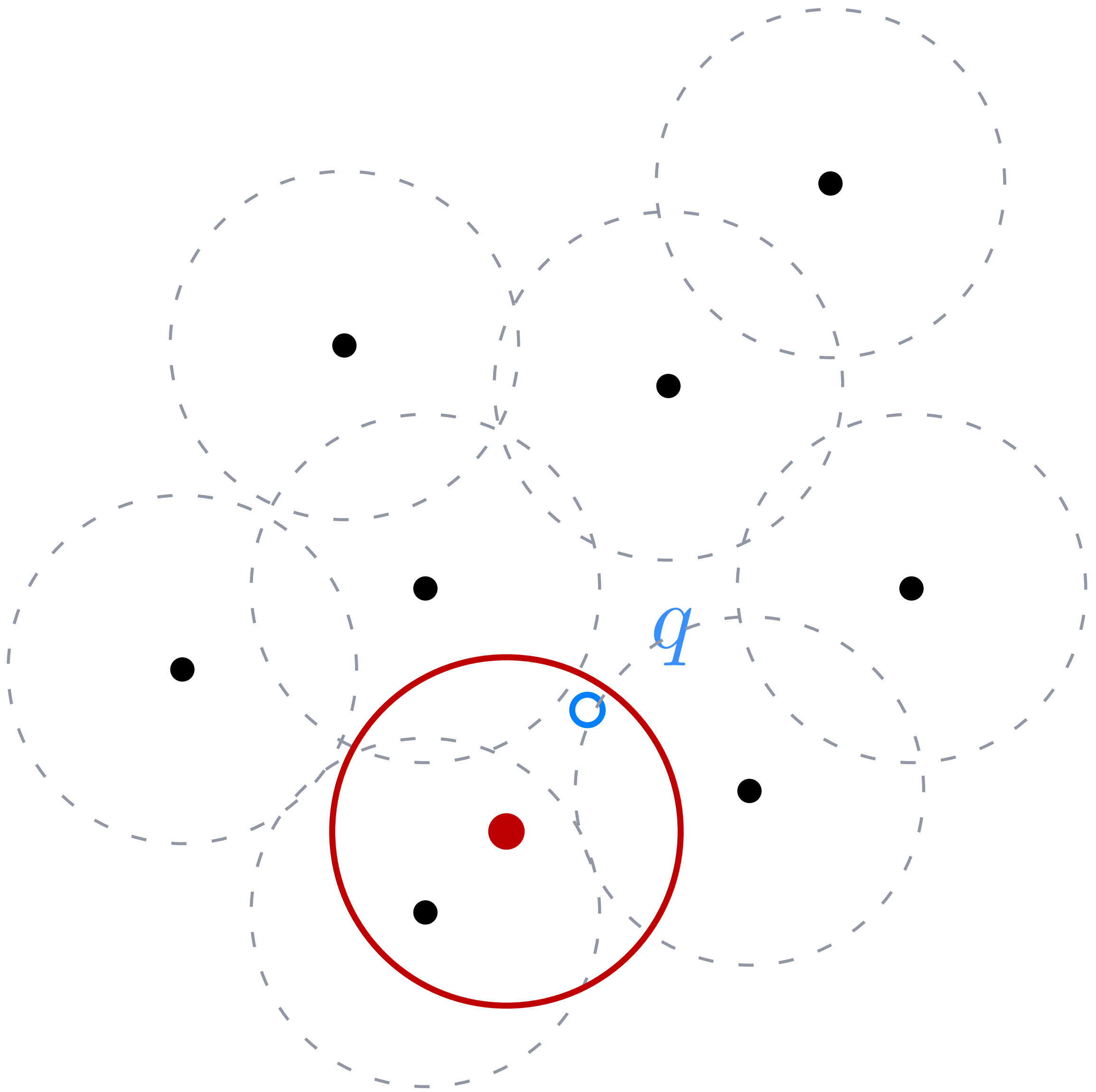
A query can be resolved by iteratively checking for each ball in $\mathcal{U}(P, r)$ if it contains q



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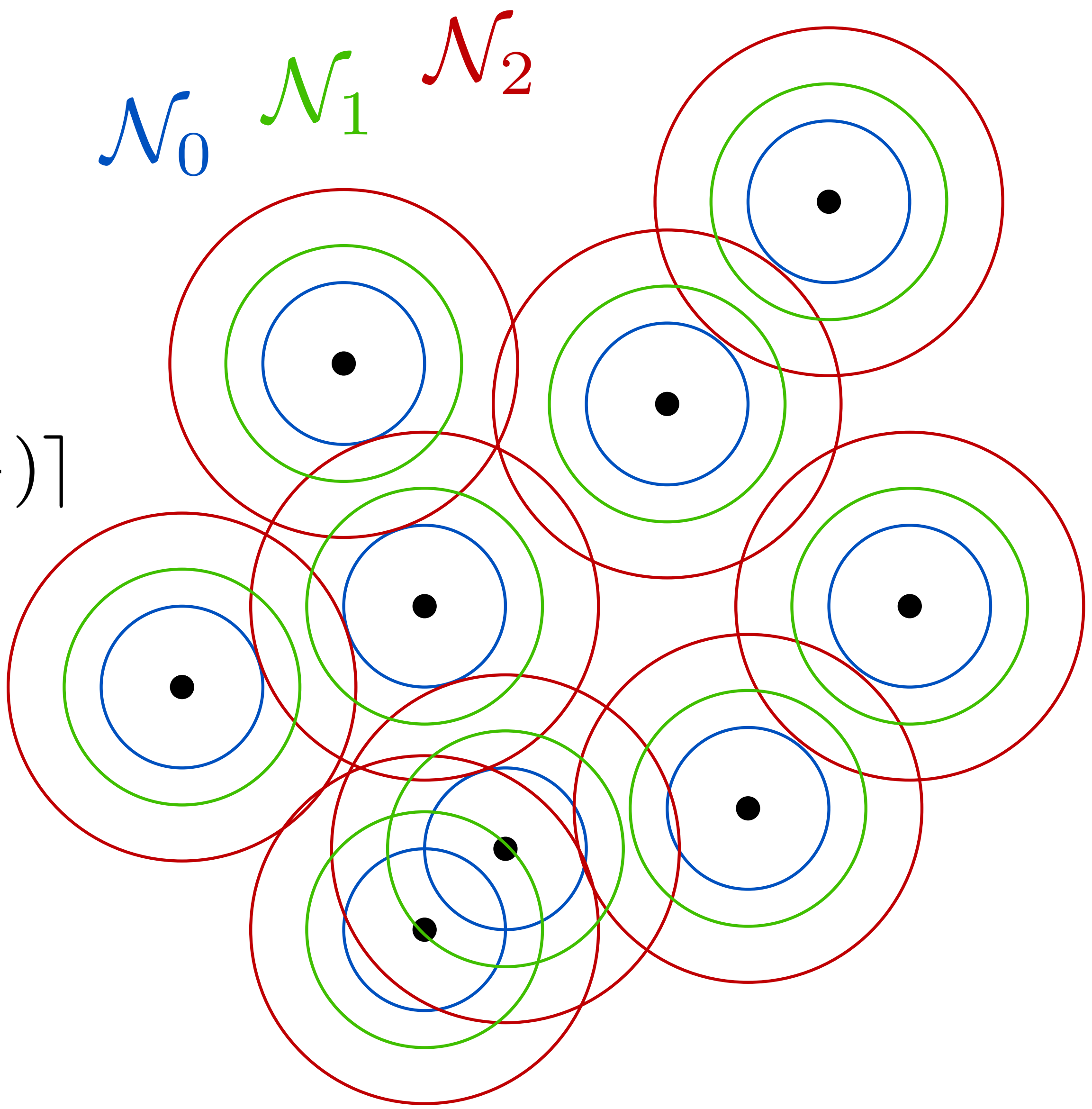

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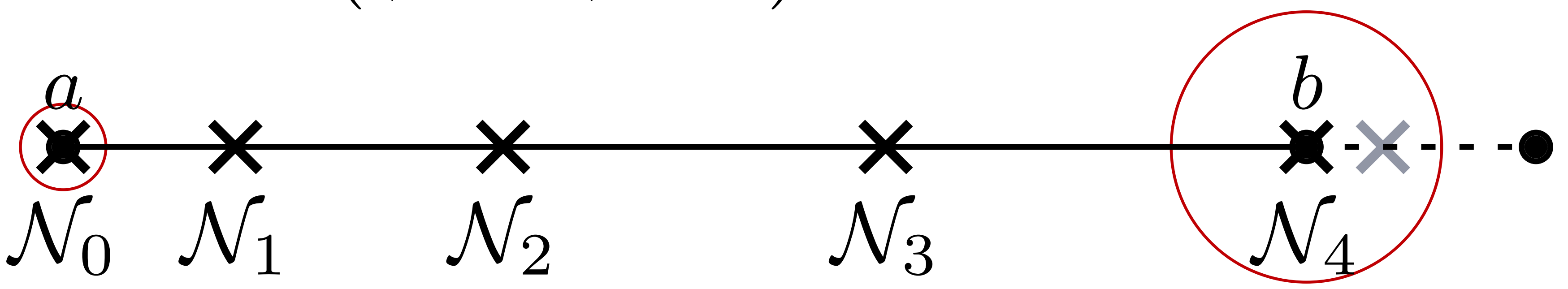

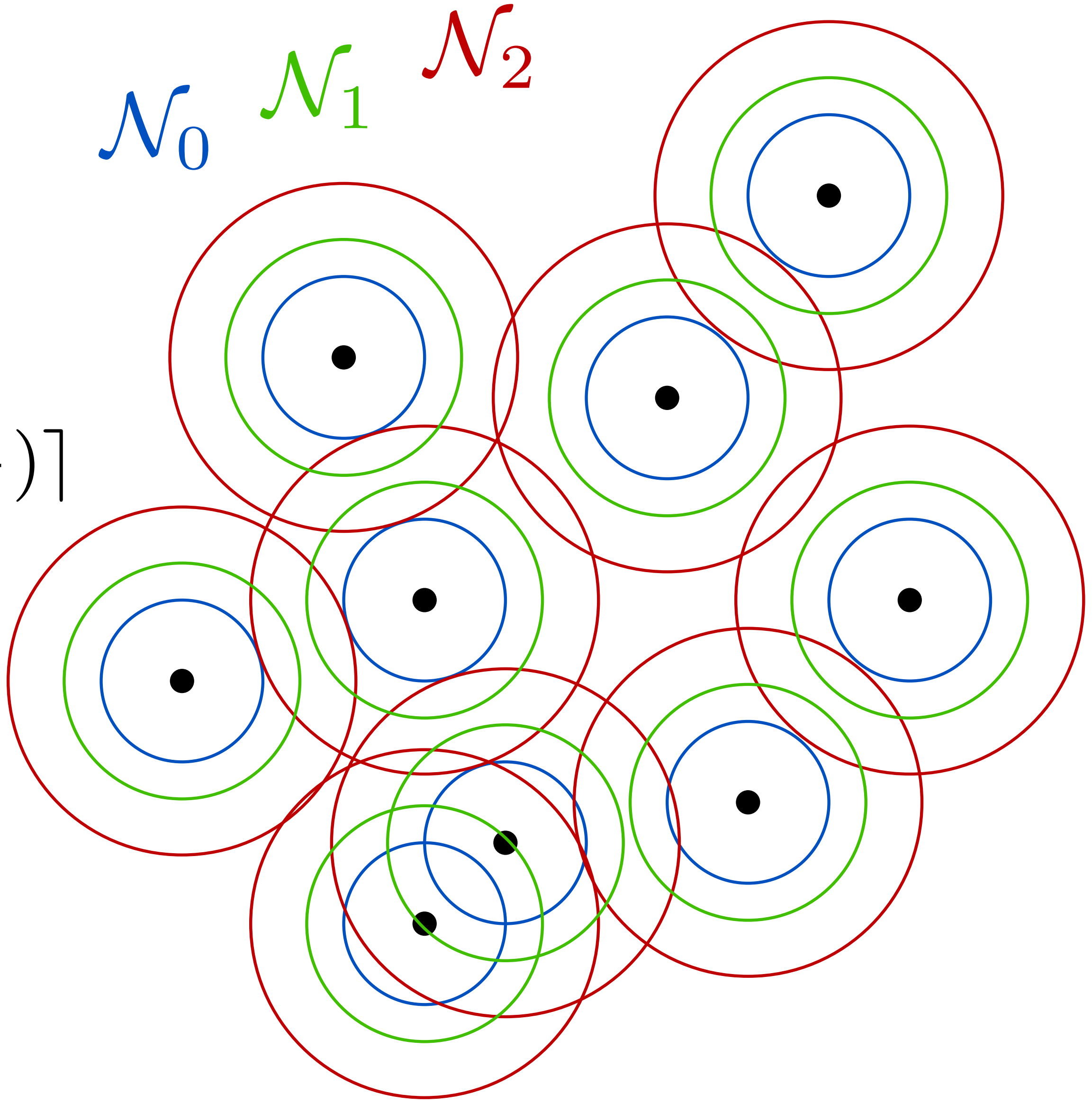

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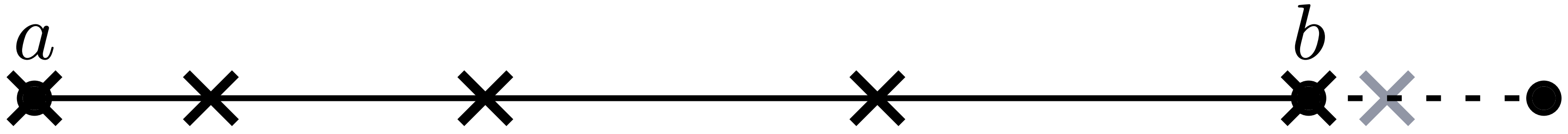
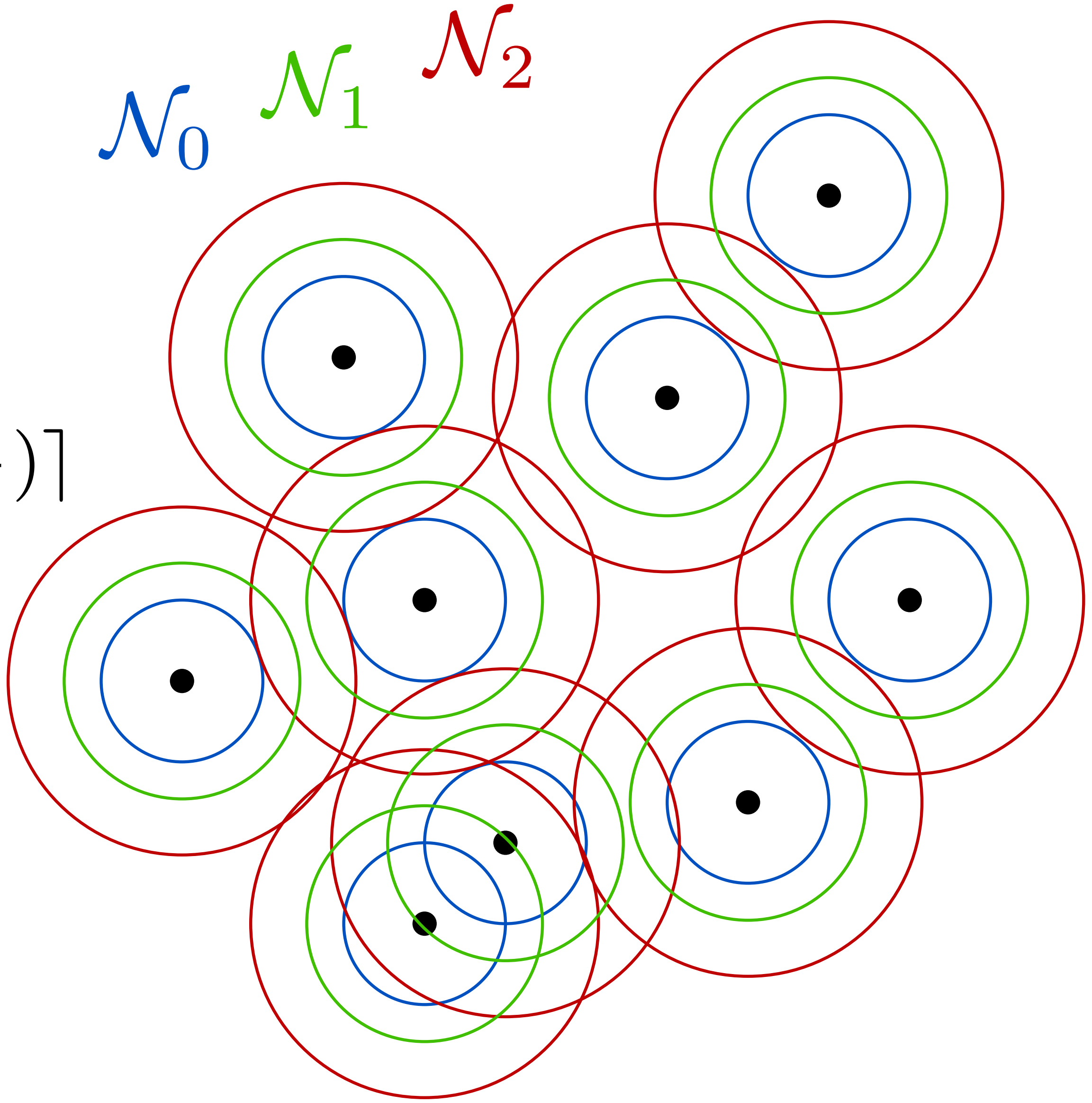


Interval near neighbor data structure $\hat{\mathcal{I}}(P, a, b, \varepsilon)$

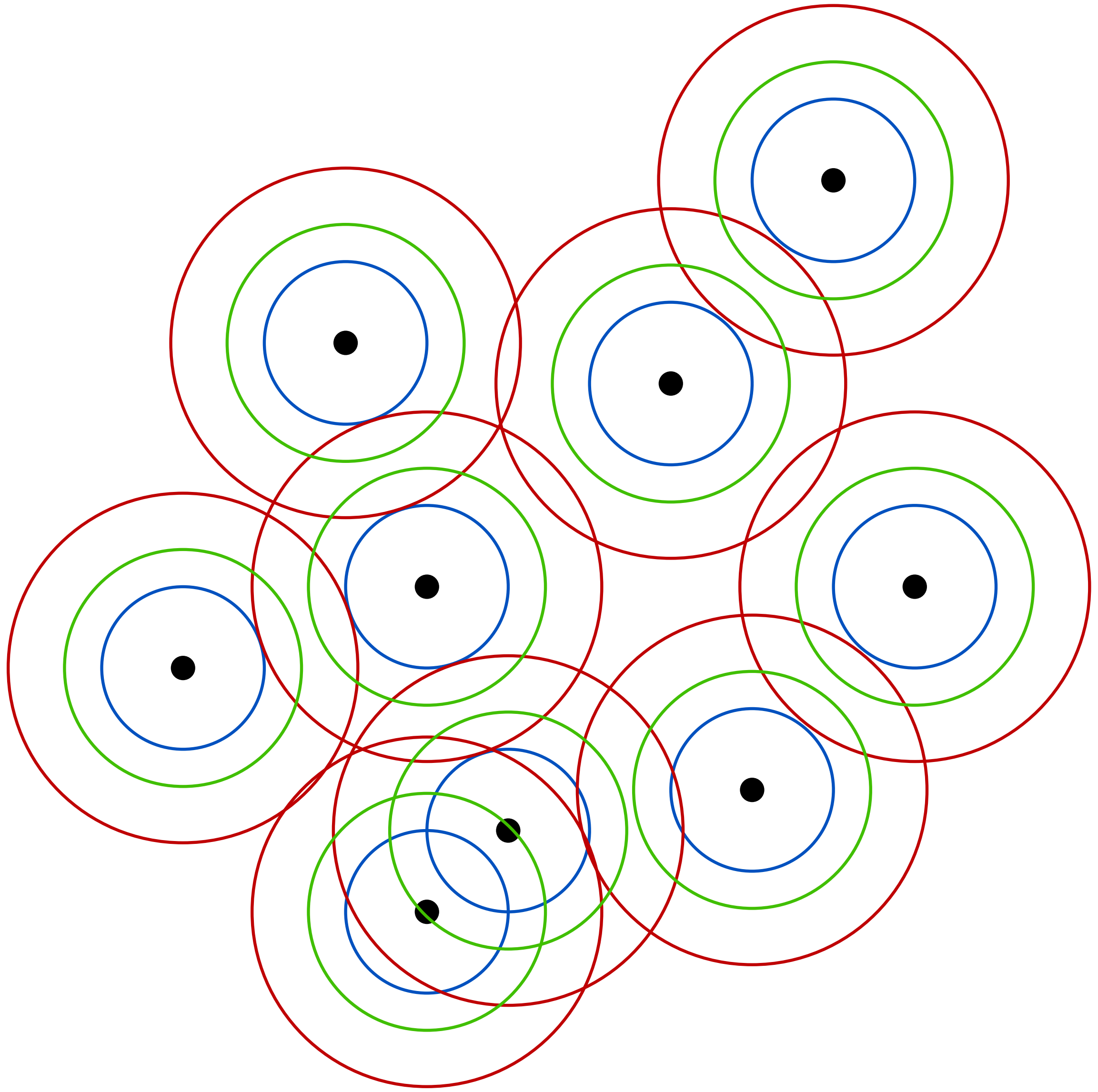
Let $\hat{\mathcal{I}}(P, a, b, \varepsilon) = \{\mathcal{N}_0, \ldots, \mathcal{N}_M\}$



Handling a range of radii

Interval near neighbor data structure $\hat{\mathcal{I}}(P, a, b, \varepsilon)$

Lemma: Given P, $a \leq b$ and $\varepsilon > 0$, one can construct $\hat{\mathcal{I}}(P, a, b, \varepsilon)$ such that: (A) $\hat{\mathcal{I}}$ is made out of $O(\varepsilon^{-1} \log(b/a))$ nn structures, and (B) given a query point q it can decide if:



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1. $d(q, P) \leq a$, *or*
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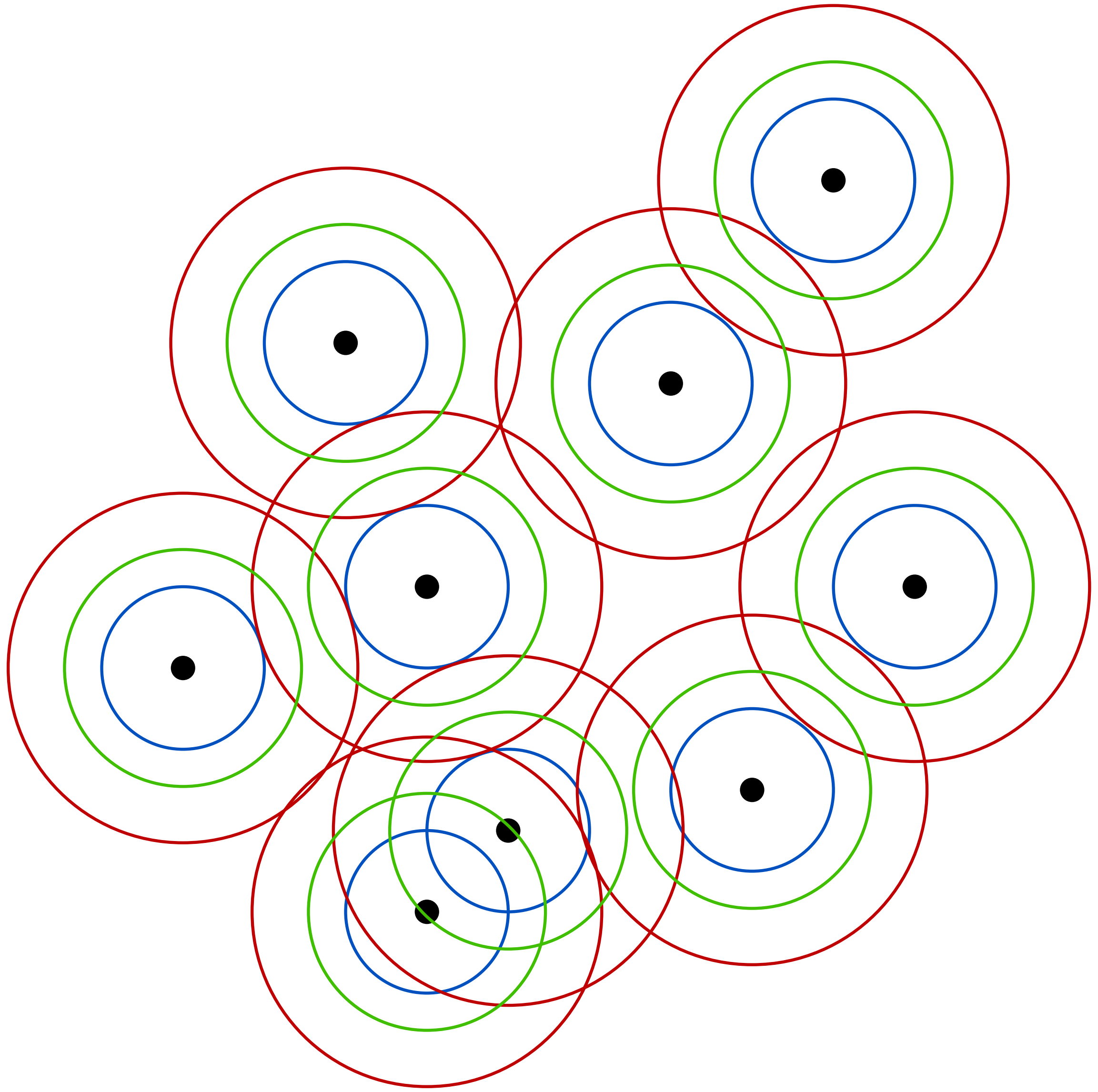
3. *Return a radius* r *and point* p *s.t.*
   $d(q, P) \leq r = d(q, p) \leq (1 + \varepsilon) d(q, P)$



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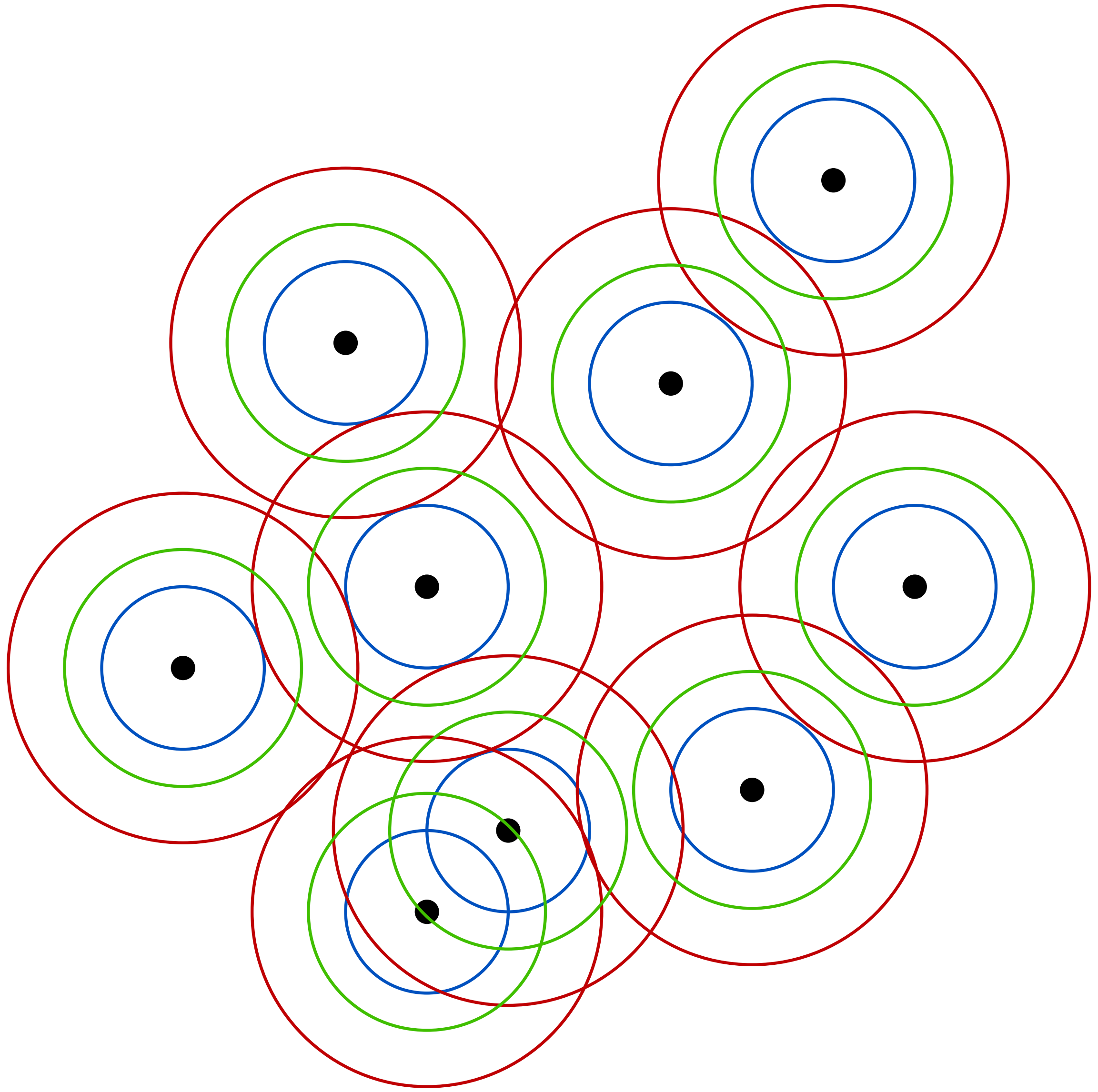
number of queries required is $O(\log(\varepsilon^{-1} \log(b/a)))$



Handling a range of radii

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Recall, $M = \lceil \log_{1+\varepsilon}(b/a) \rceil$

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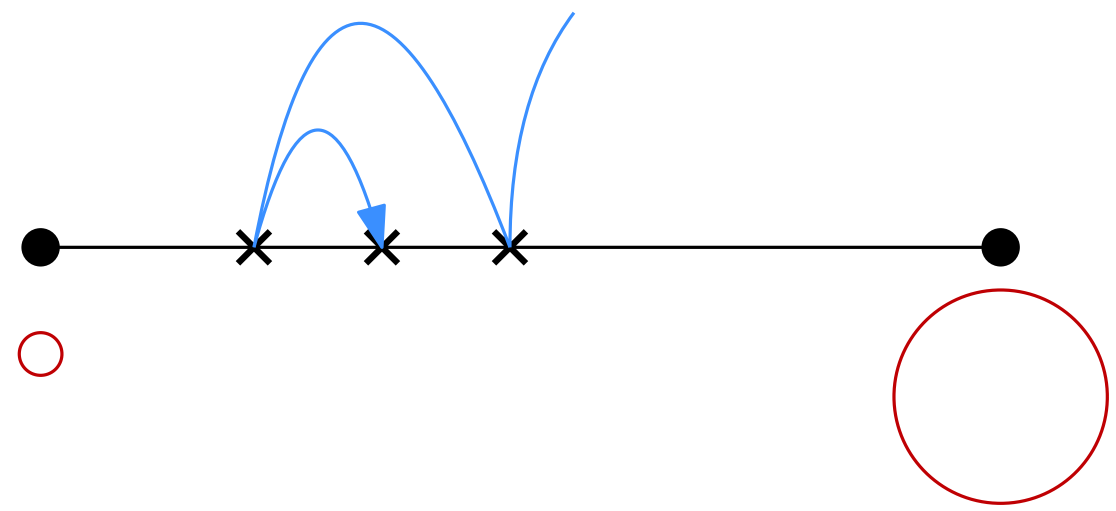
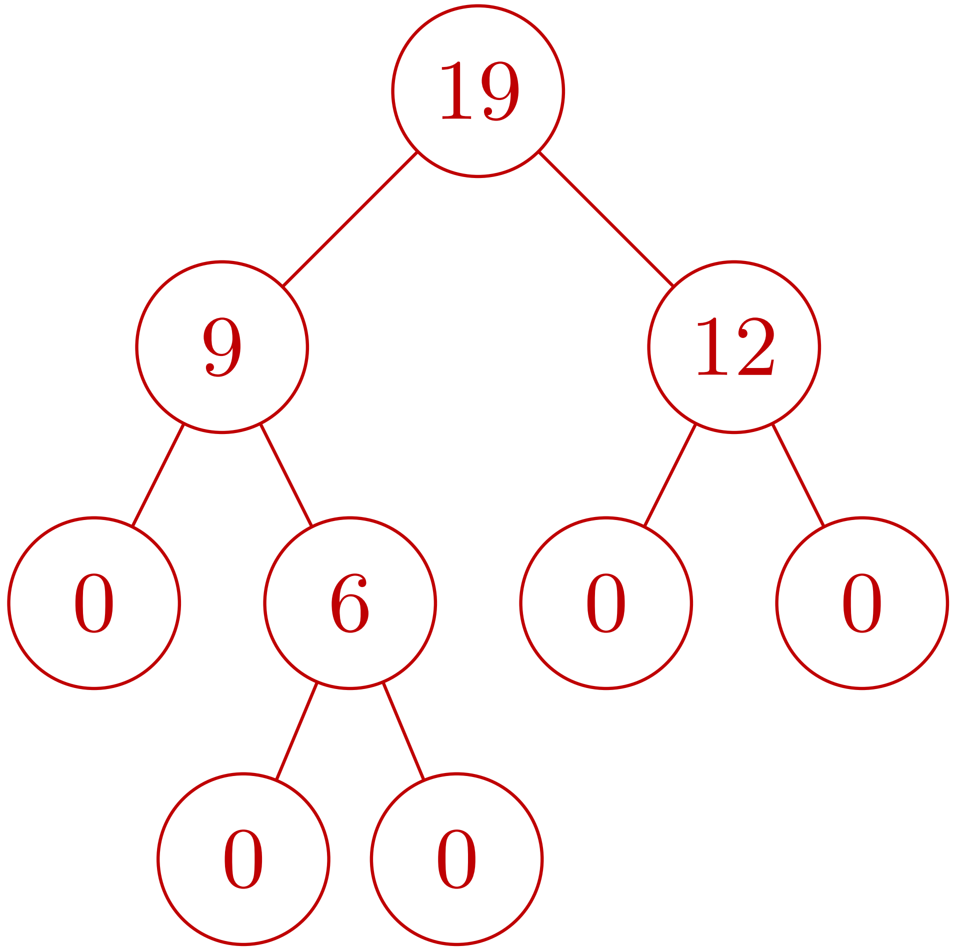
$O(\log(b/a)/\varepsilon)$ nn structures

number of queries can be achieved by doing a binary search on the radius:
$O(\log(\varepsilon^{-1} \log(b/a)))$



The ANN data structure

Given: set of points P and
a t-approximate (B)HST H on P



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Given: set of points P and
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recall:

- Each vertex v has a label $\Delta_v \geq 0$.
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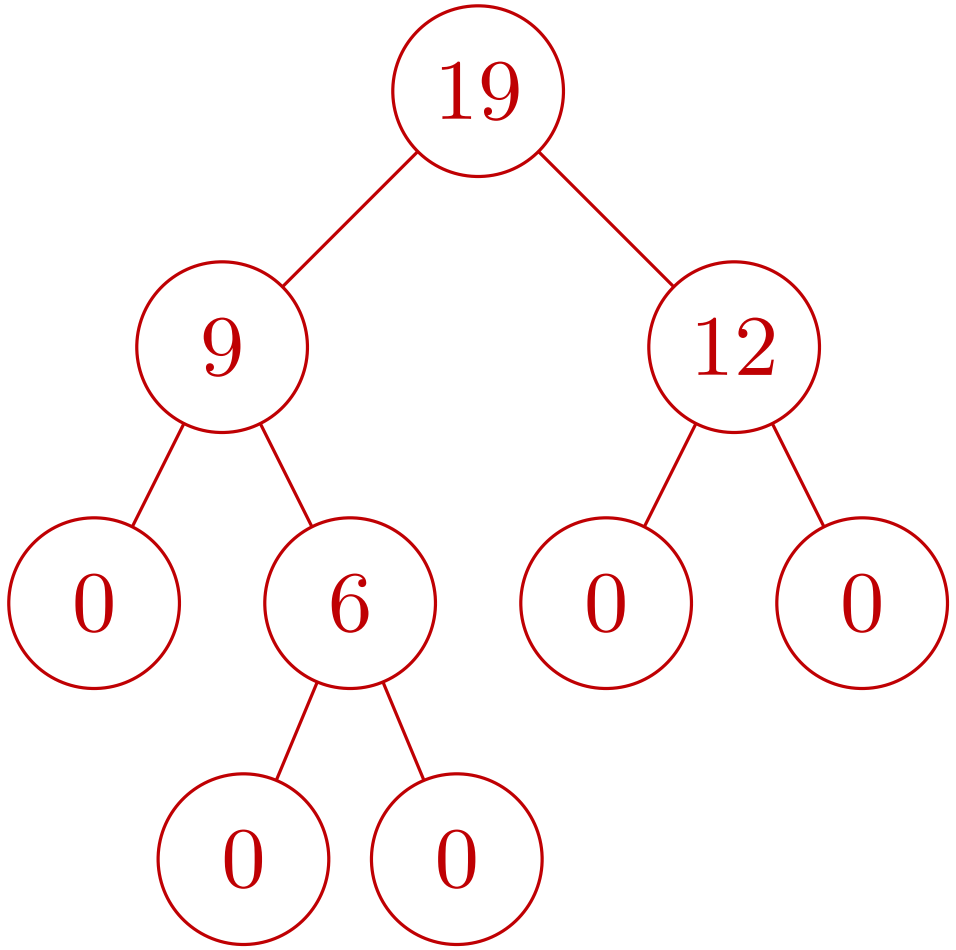
- If u is a child of v, then $\Delta_v \geq \Delta_u$
- $\Delta_{lca(u,v)}$ denotes the t-approximated distance between two leaves u and v



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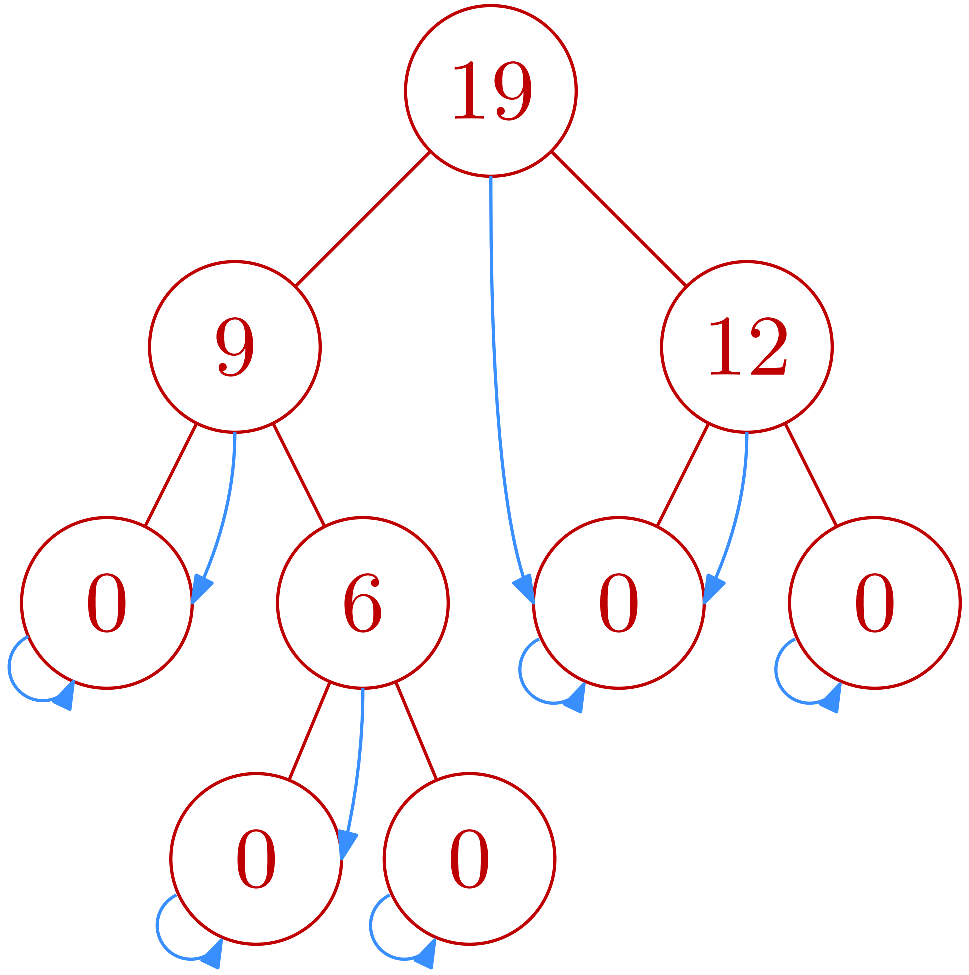
- Each vertex v has a representative leaf repr$_v$.
- repr$_u \in \{$repr$_v \mid$ v is a child of u$\}$



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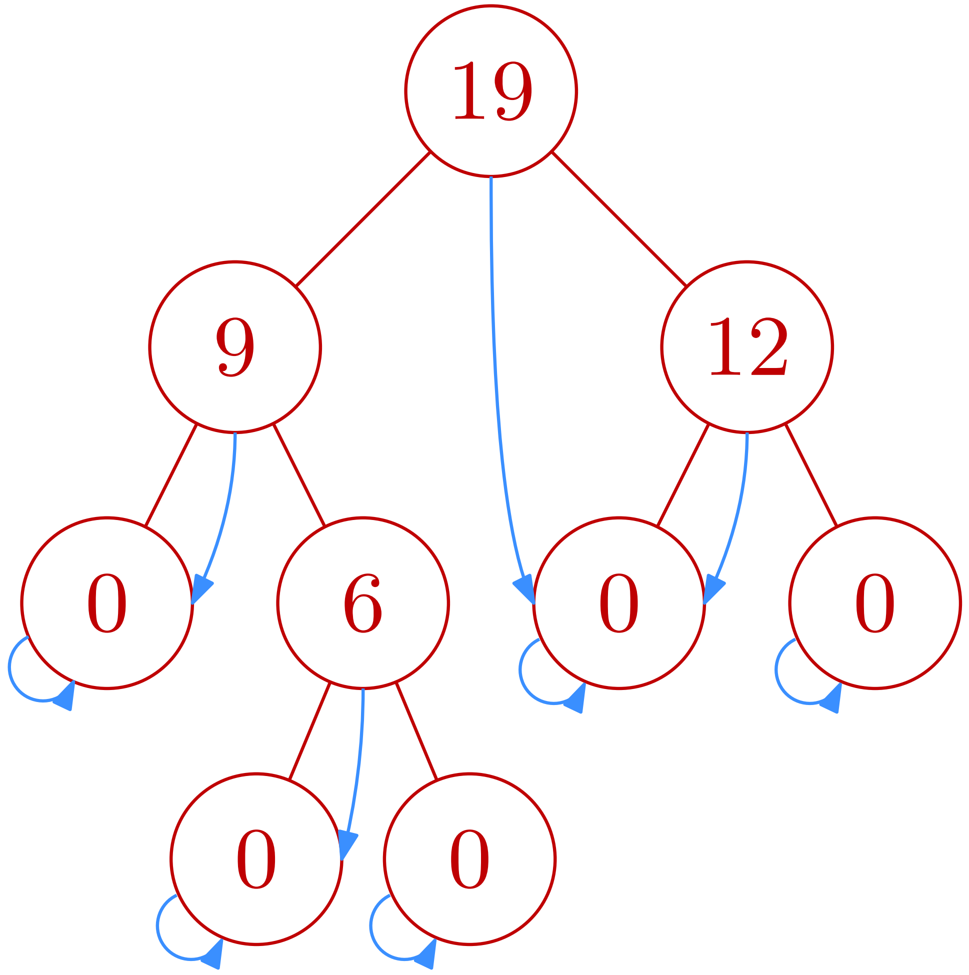
Recursively build search tree T (top-down):



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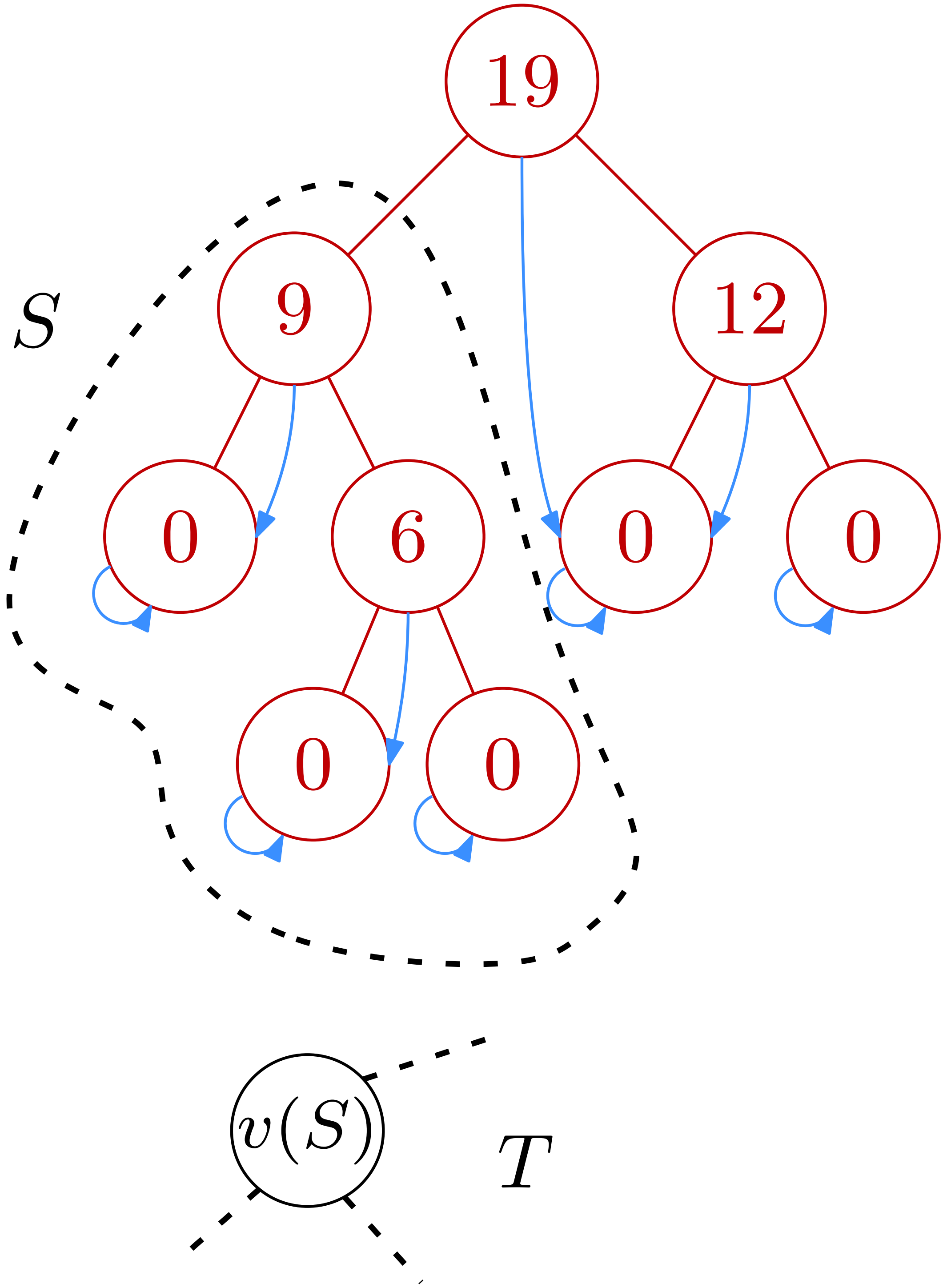

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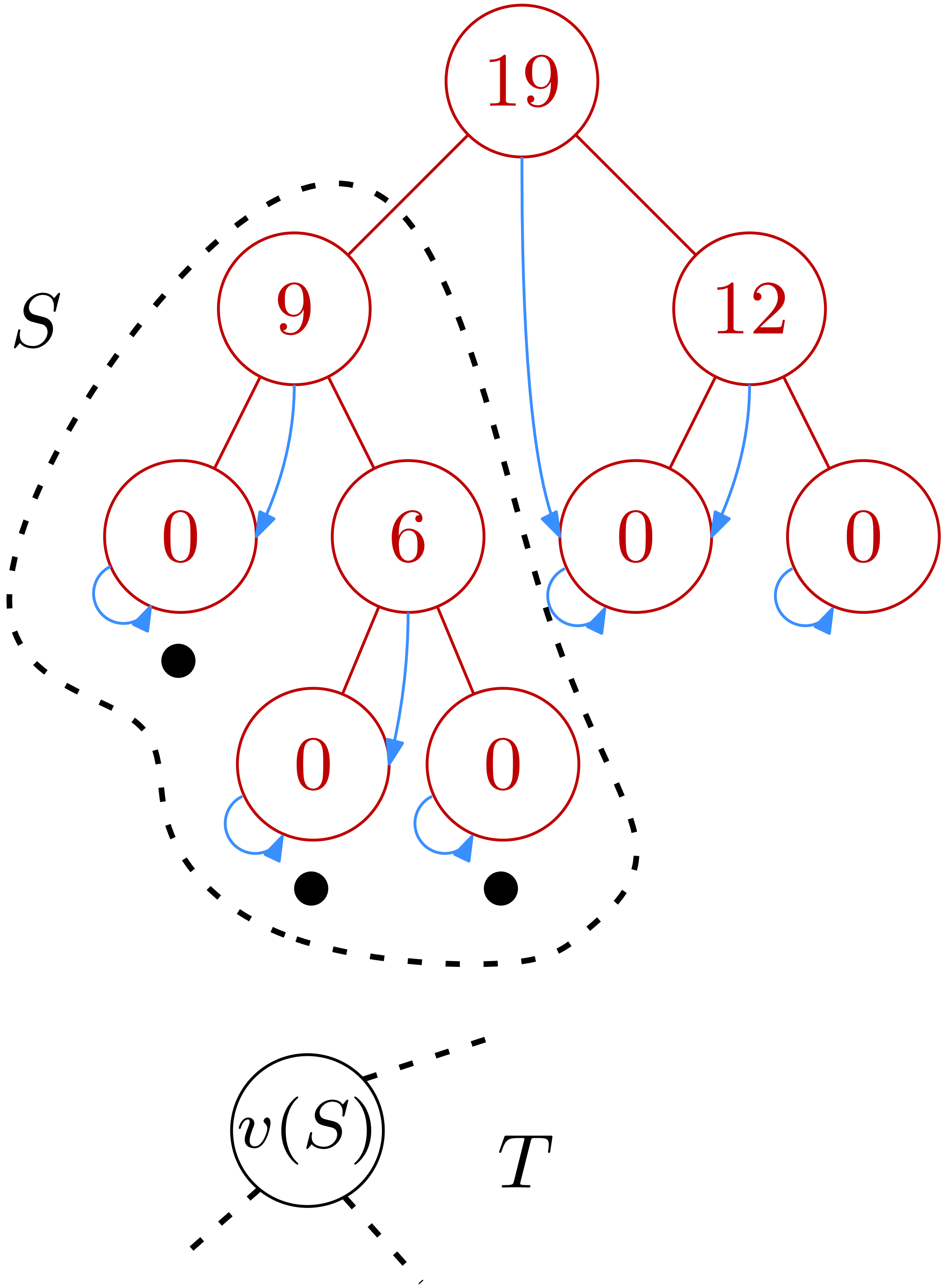

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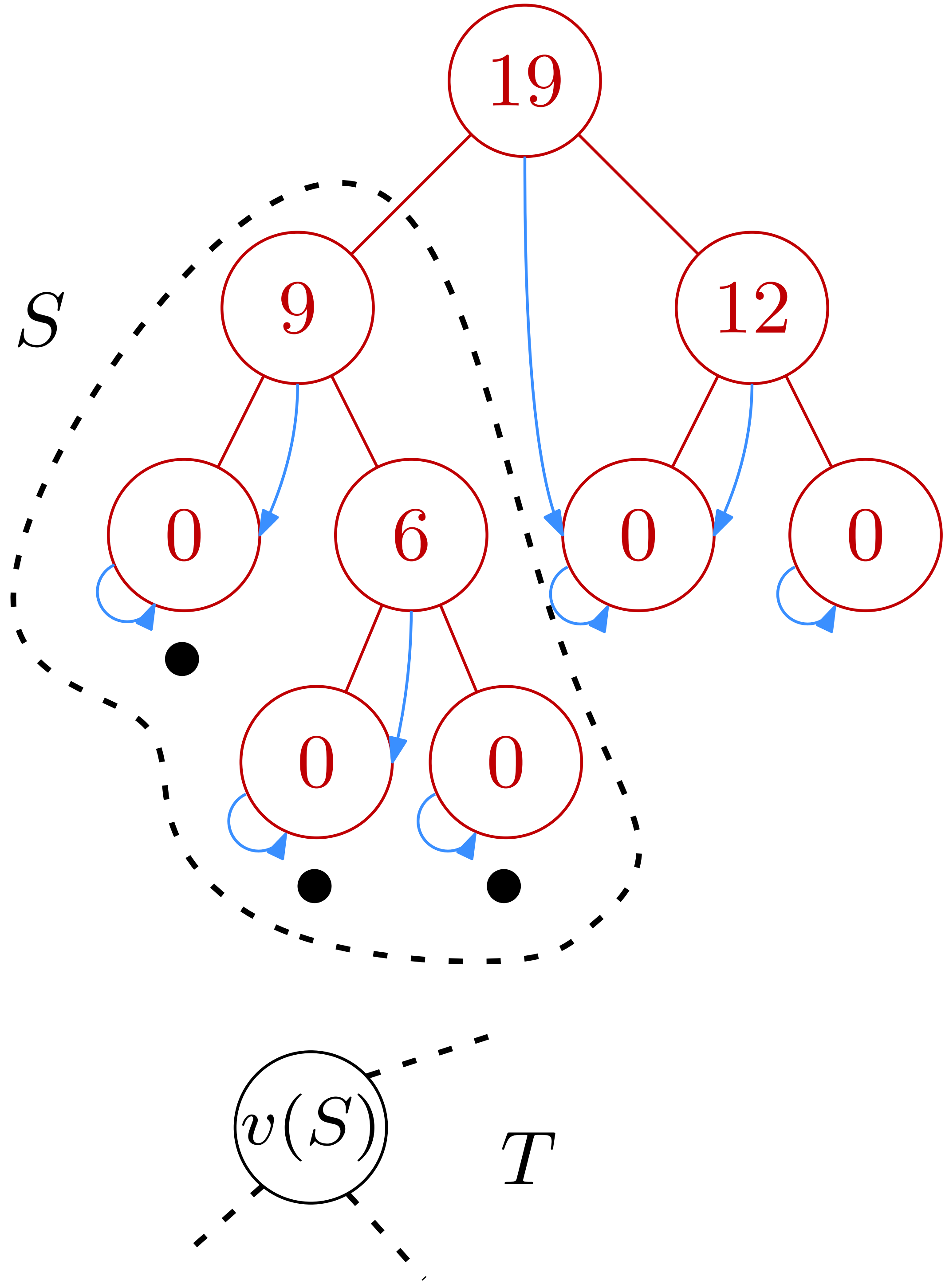

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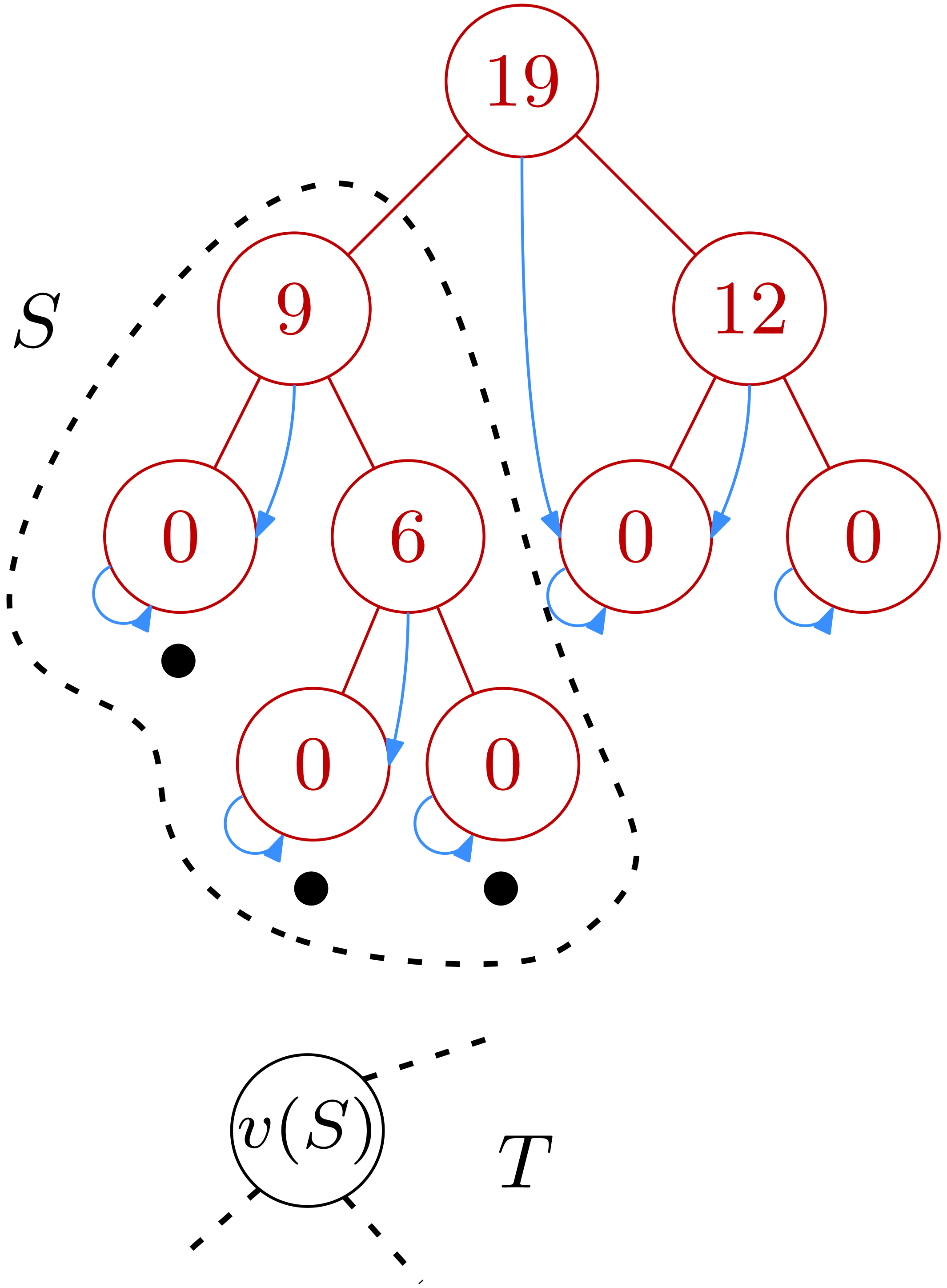
Removing a separator node breaks a (sub)tree T into connected components of size at most $|V(T)|/2$



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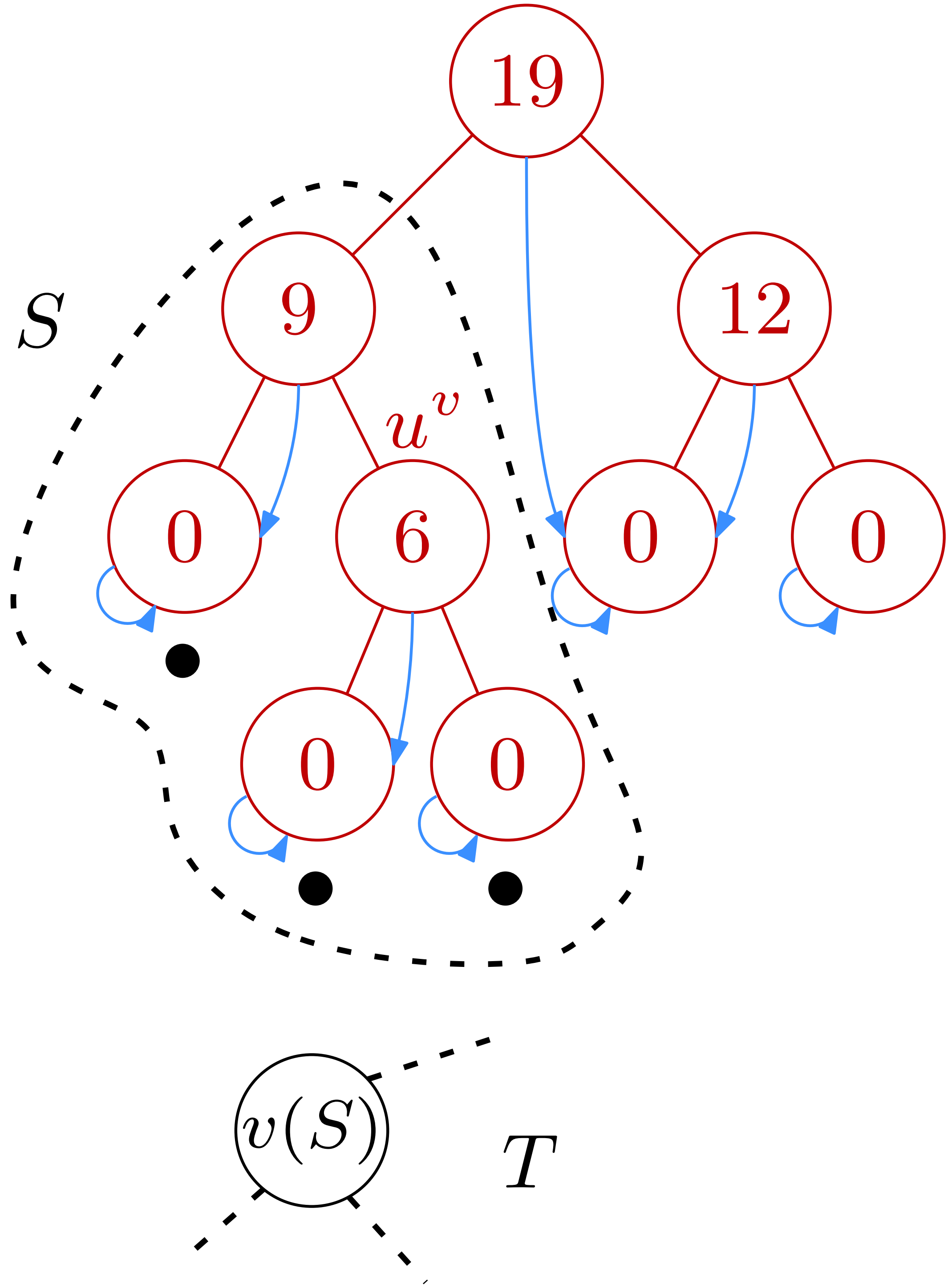

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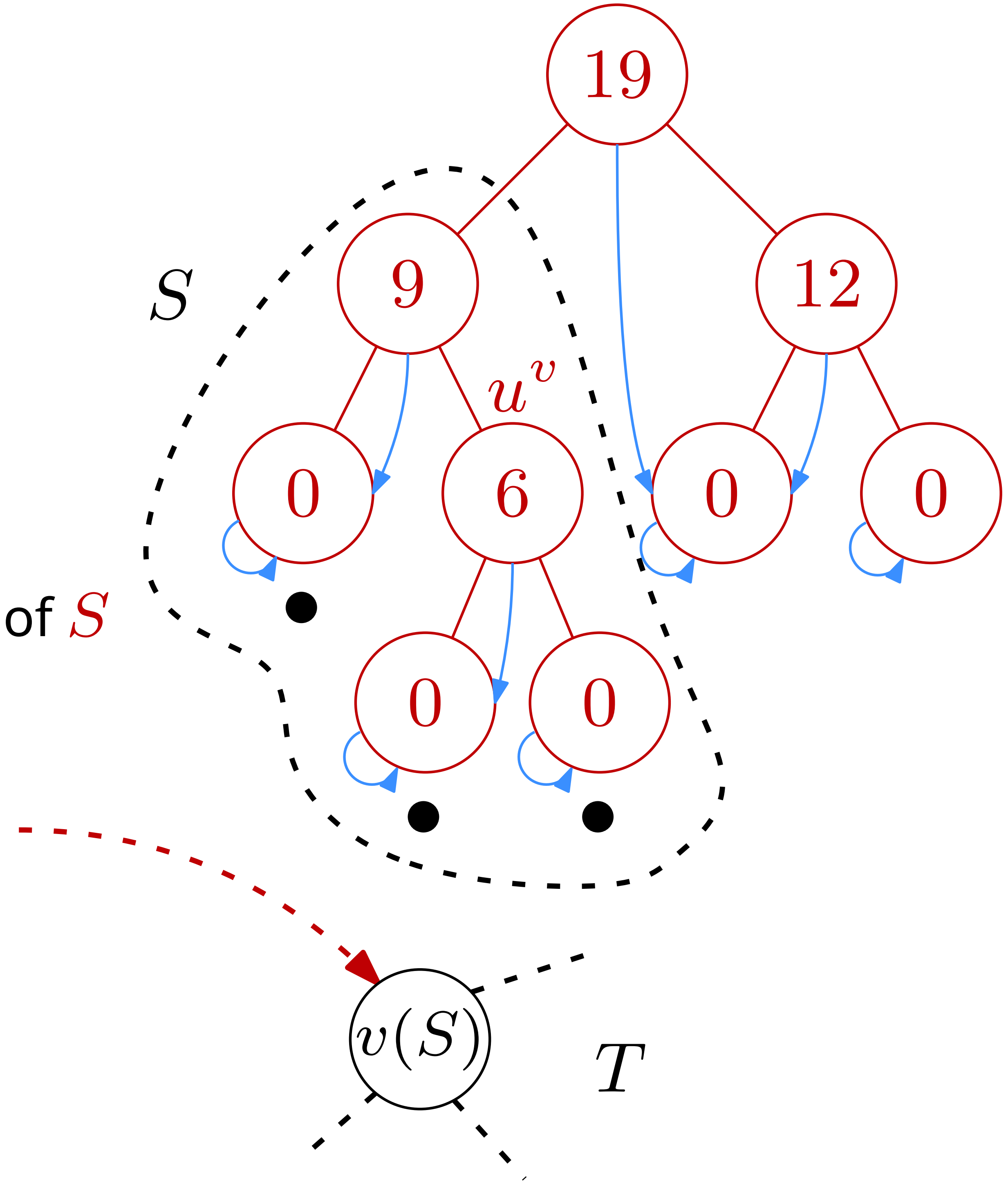

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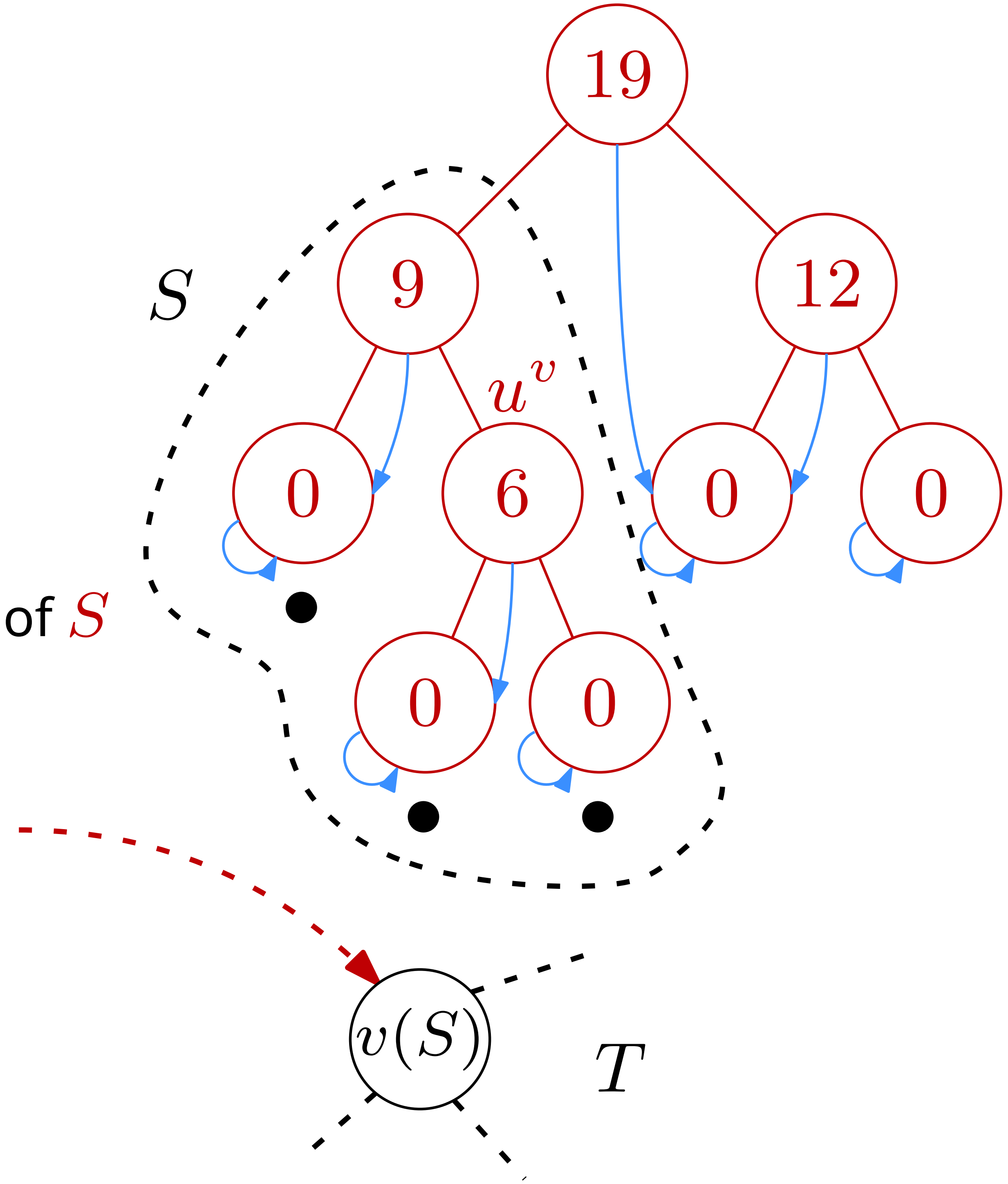
$$r_v = \frac{\Delta(u^v)}{4t} \text{ and } R_v = \mu\Delta(u^v)$$



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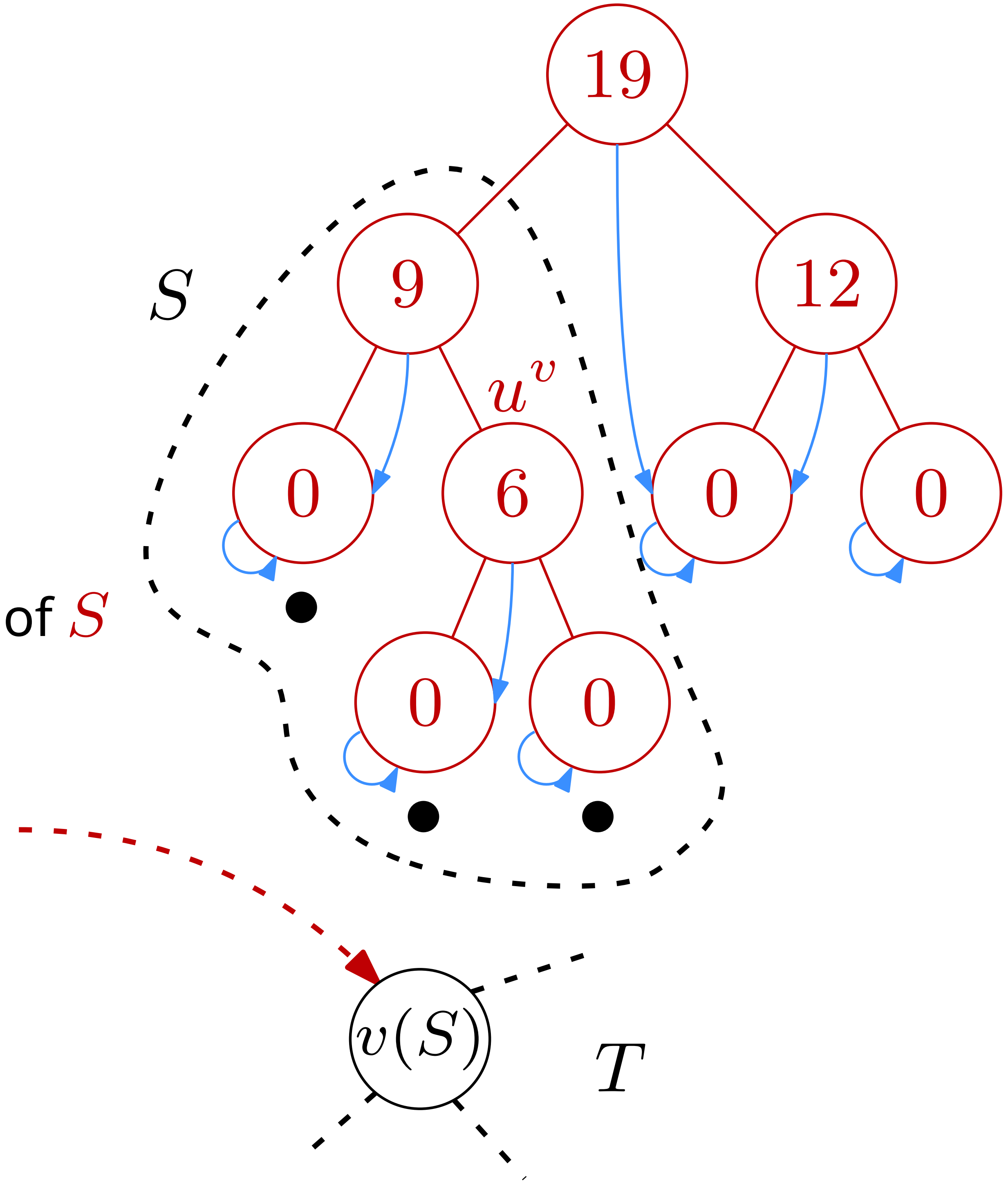

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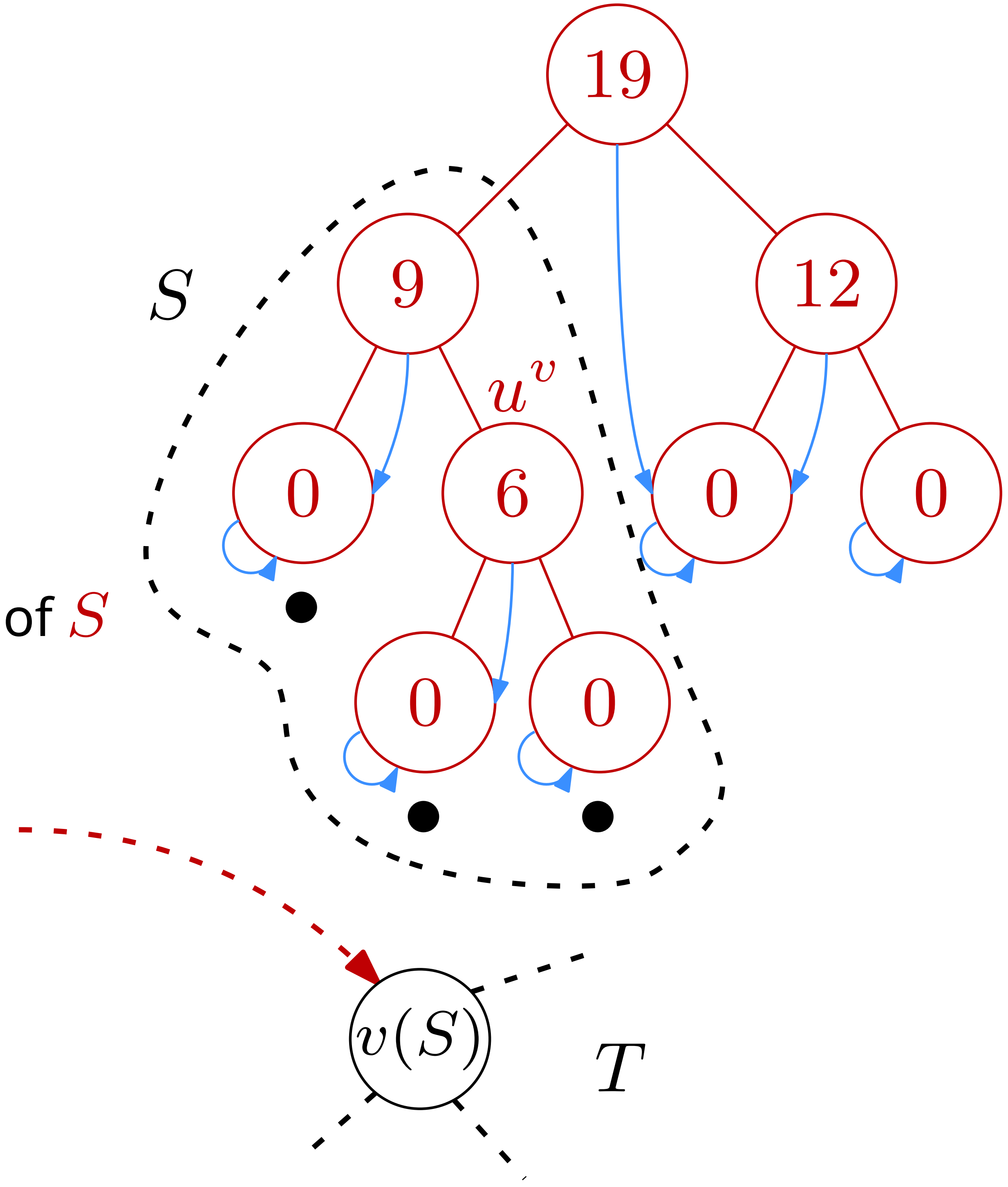
$\hat{\mathcal{I}}_v$ can be used to determine search path in T



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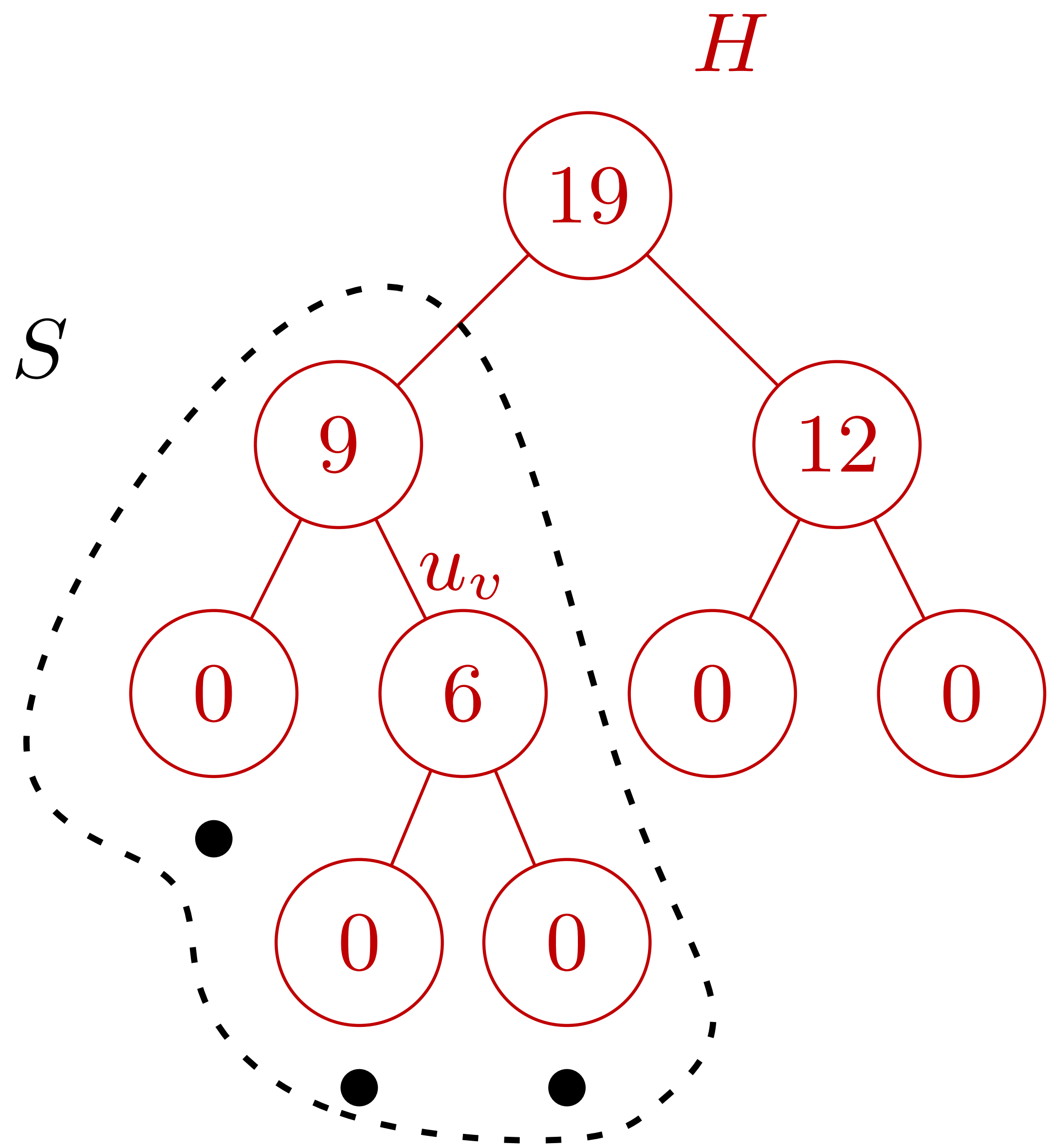

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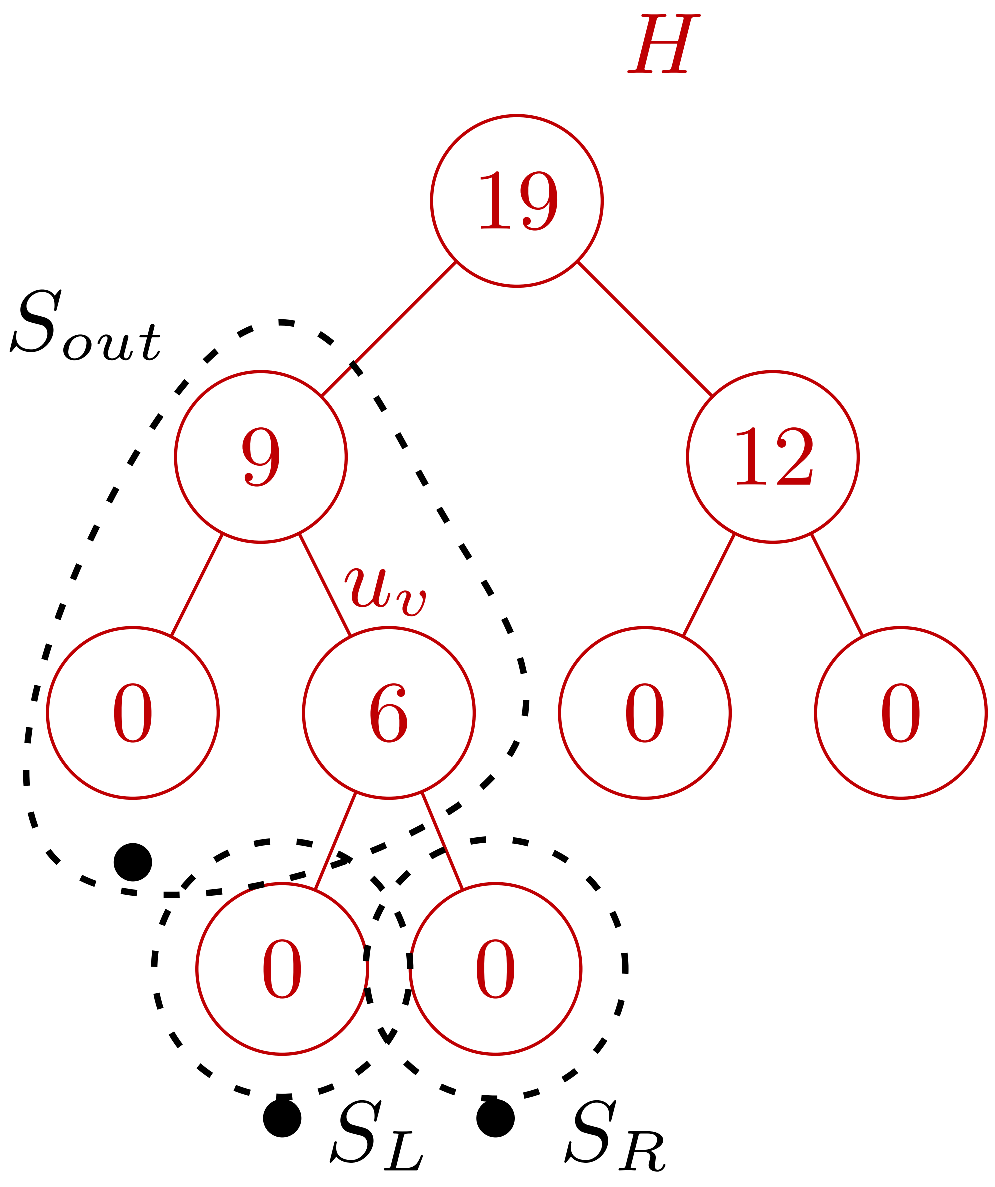
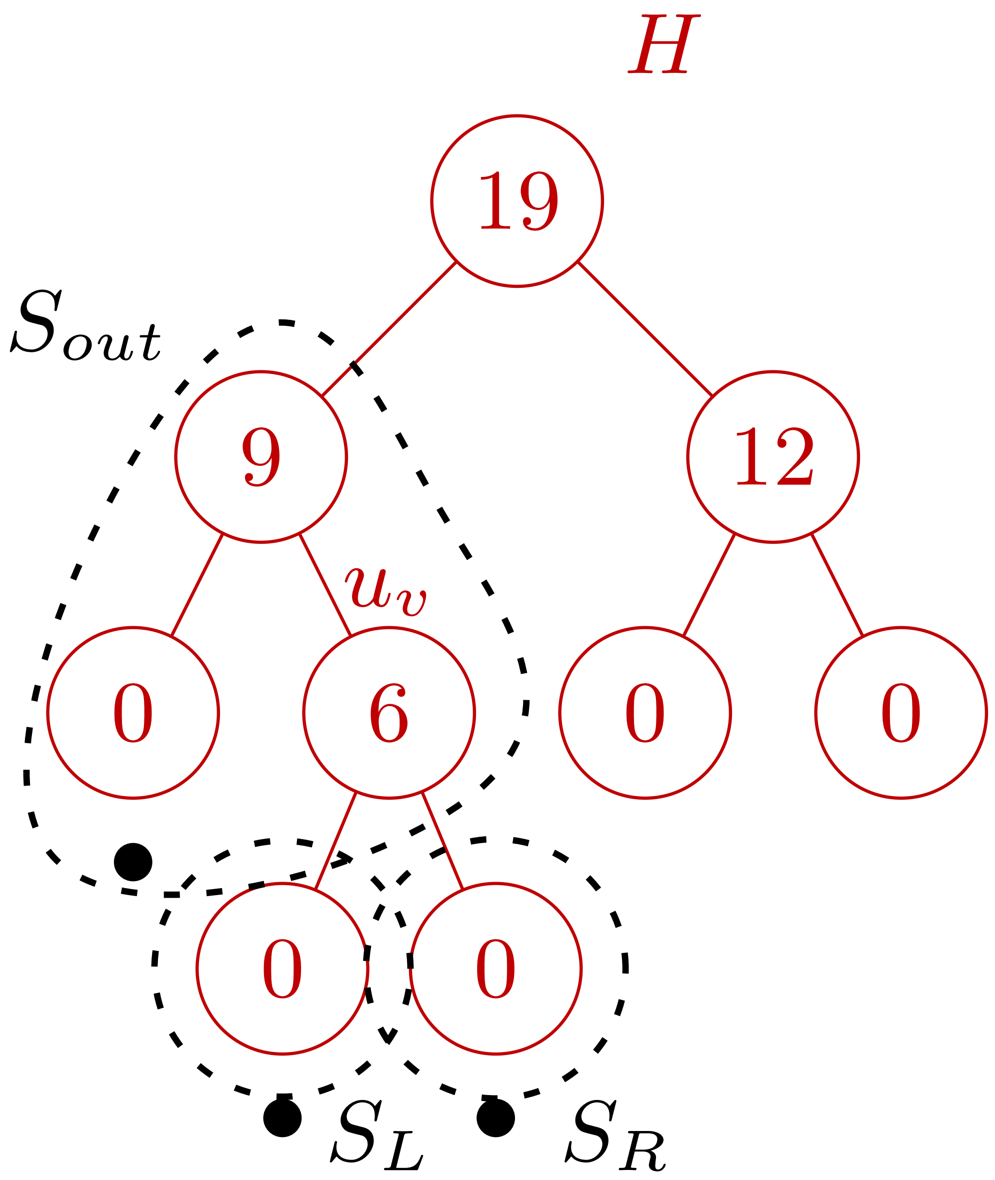


The ANN data structure

subtle:

S_{out} contains rep_{u_v} but no other points of S_L and S_R.



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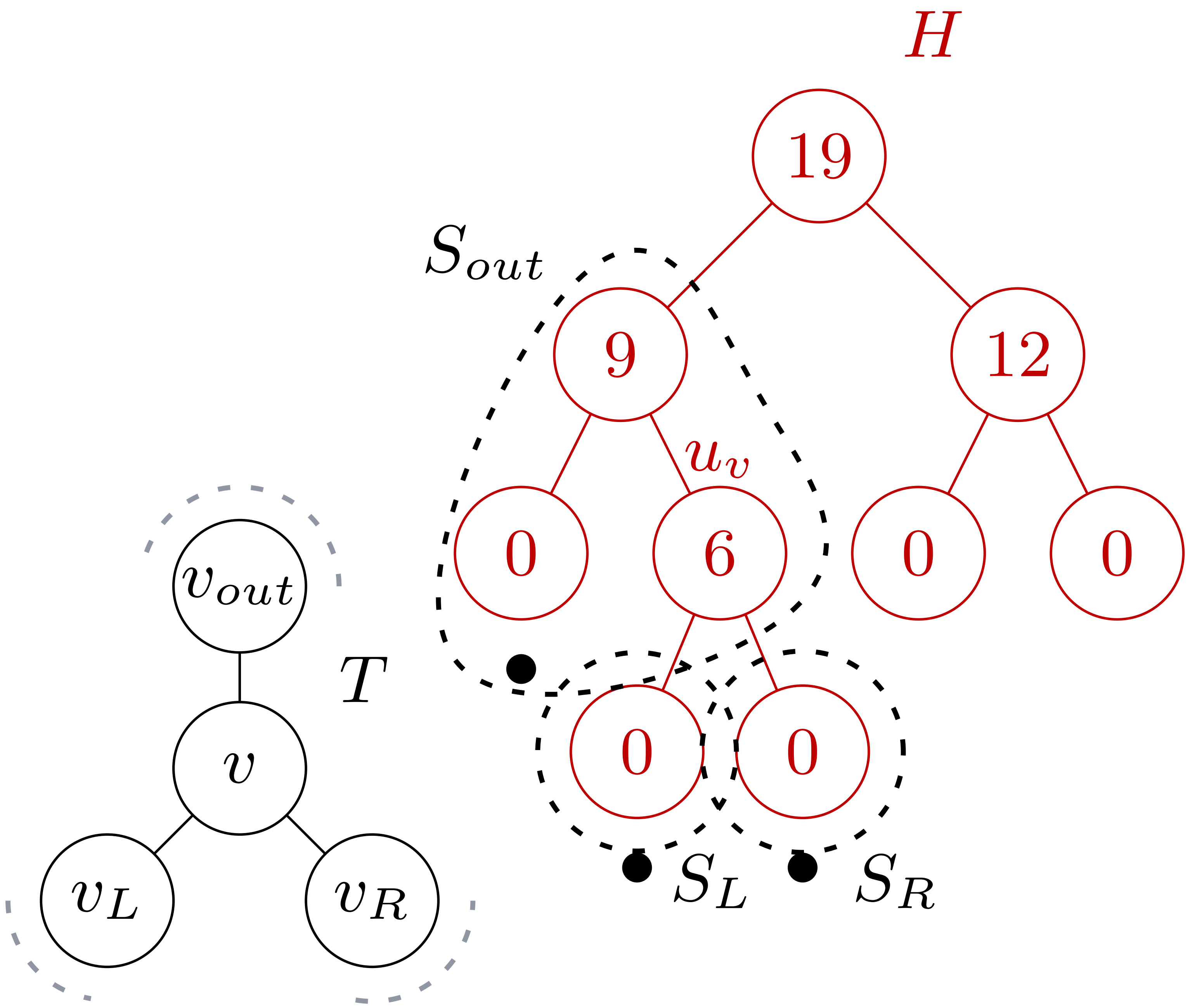

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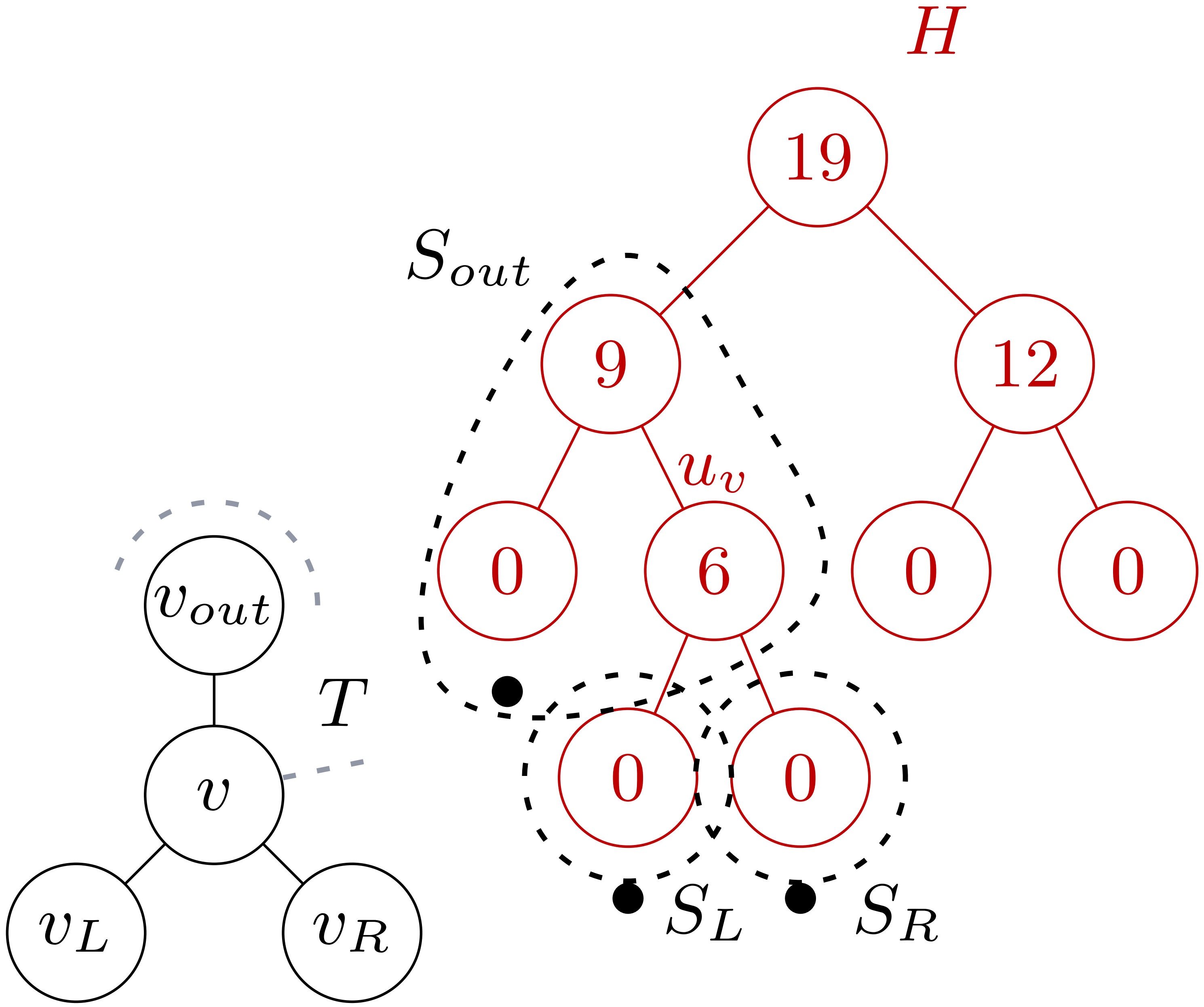

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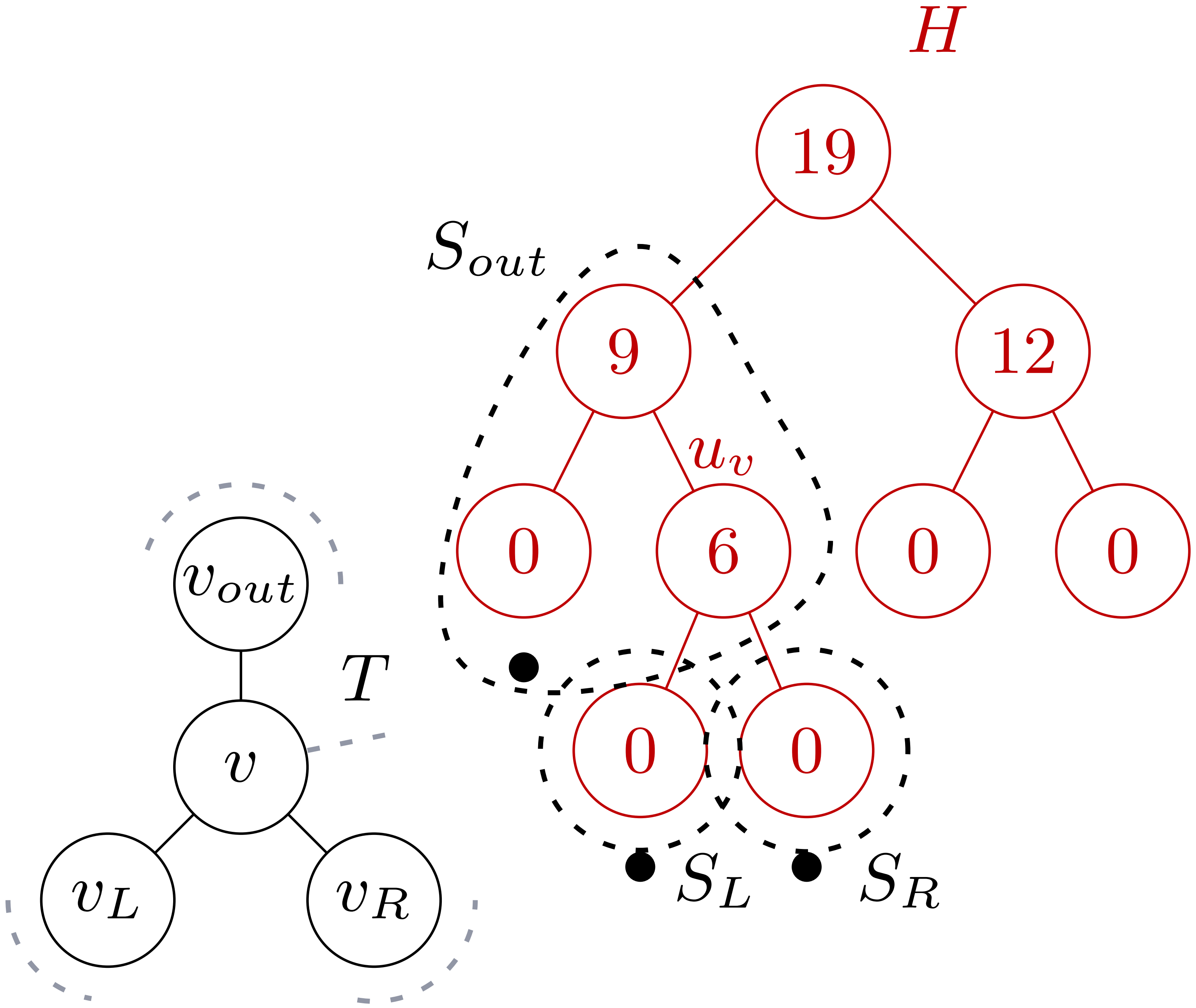
$\hat{\mathcal{I}}_v = \hat{\mathcal{I}}(P^v, r_v, R_v, \varepsilon/4)$ stored in node v

$r_v = \frac{\Delta(u^v)}{4t}$ and $R_v = \mu\Delta(u^v)$

A query into $\hat{\mathcal{I}}_v$ results in one of three cases:



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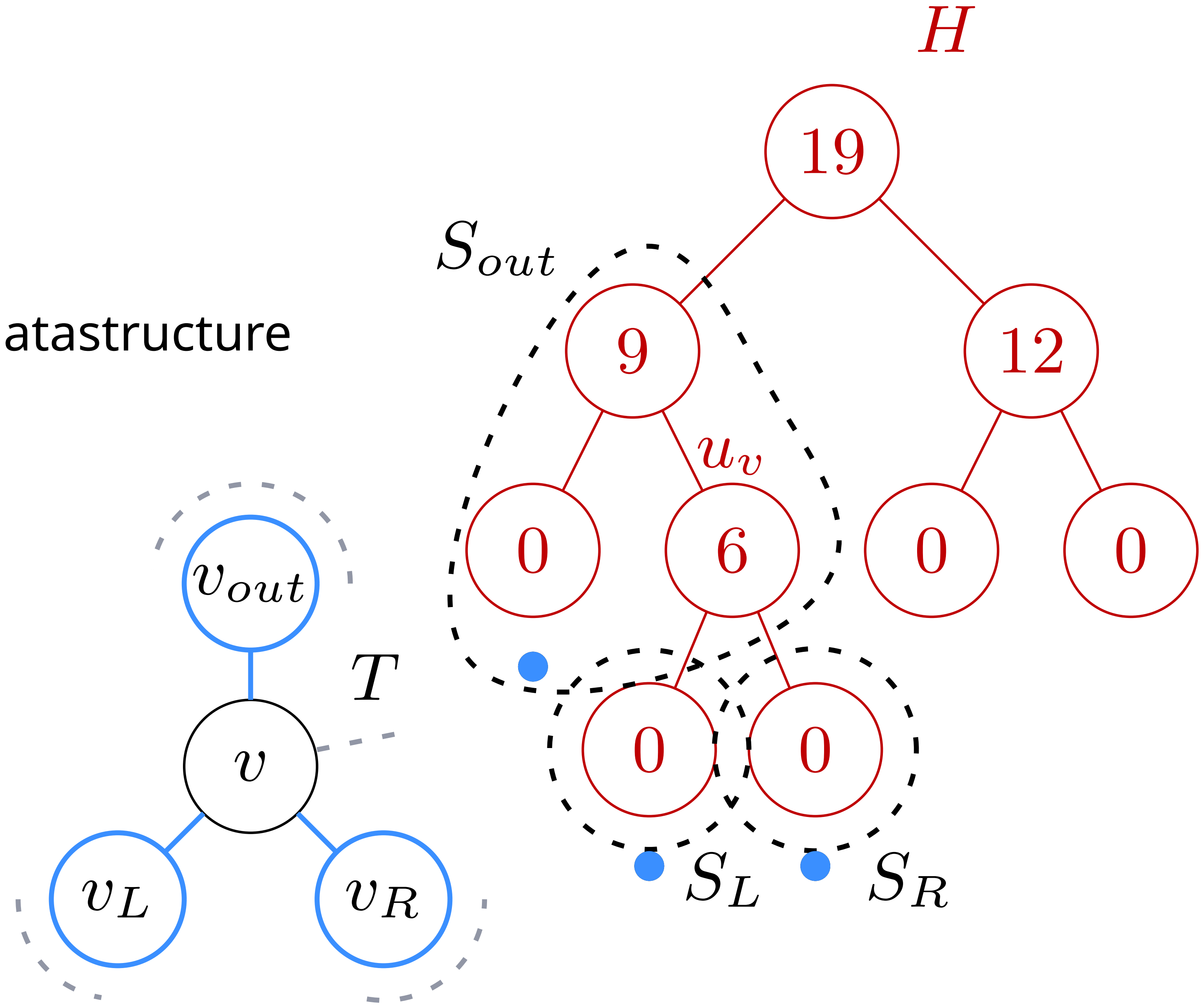
  Recurse into the subtree containing p



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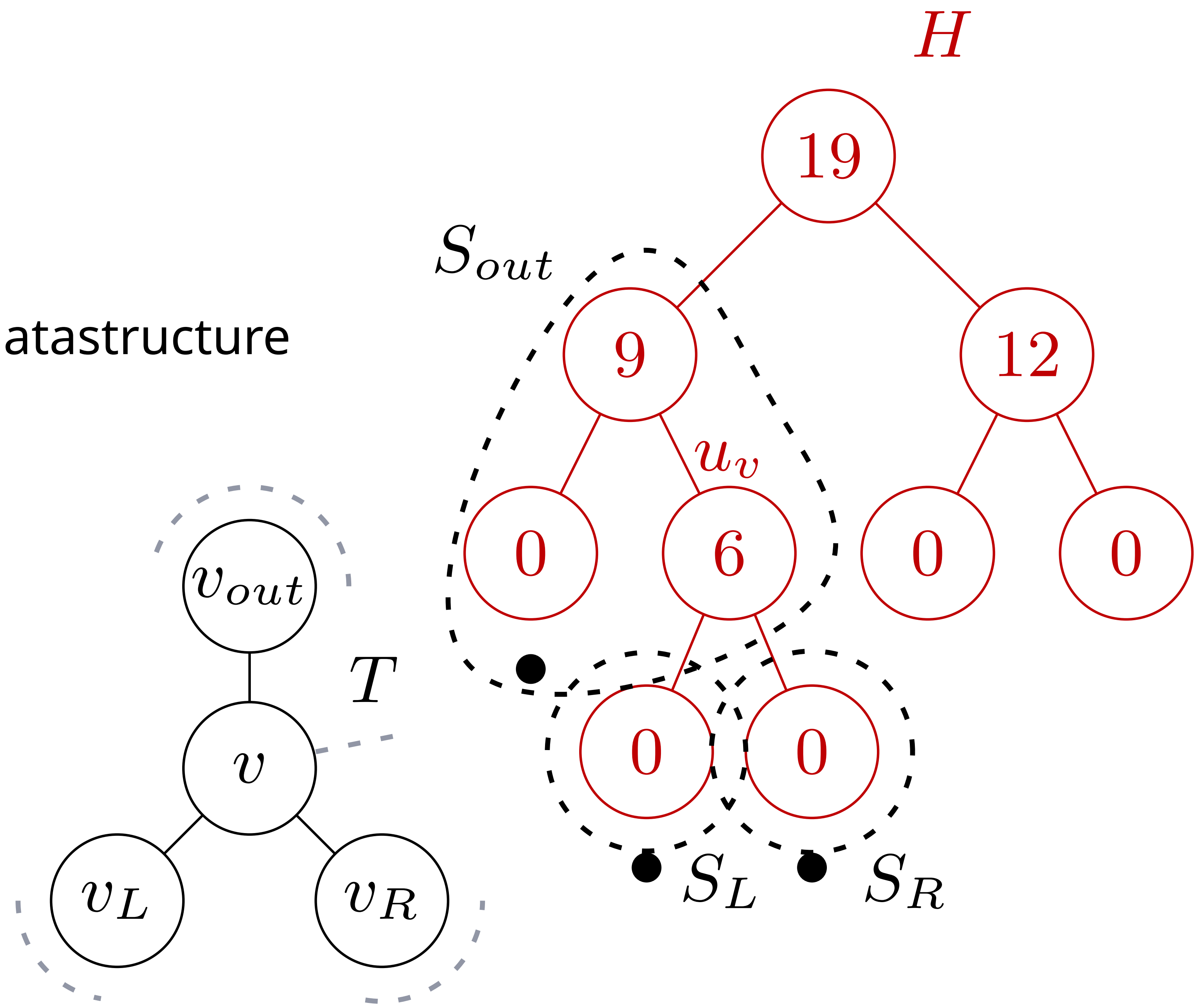
- $d(q, P^v) \in (r_v, R_v]$: Then the query finds a $(1 + \varepsilon/4)$-ANN point $s \in P^v$ and returns it as the answer to the query



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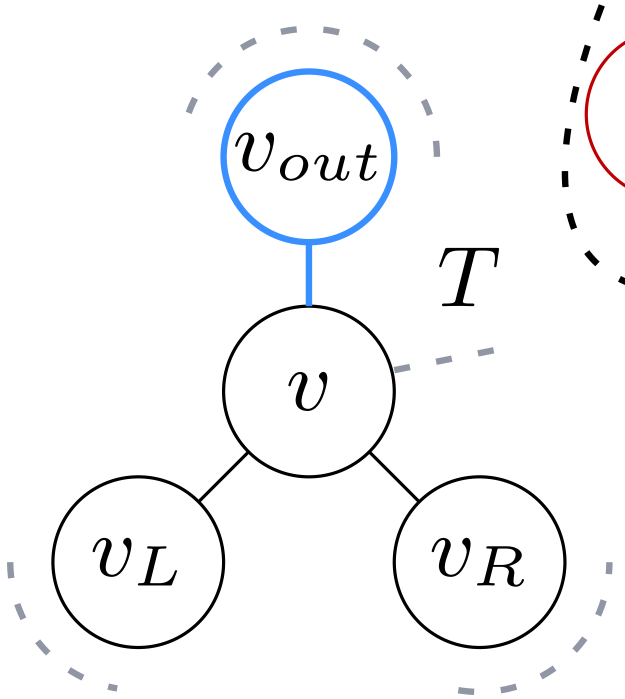
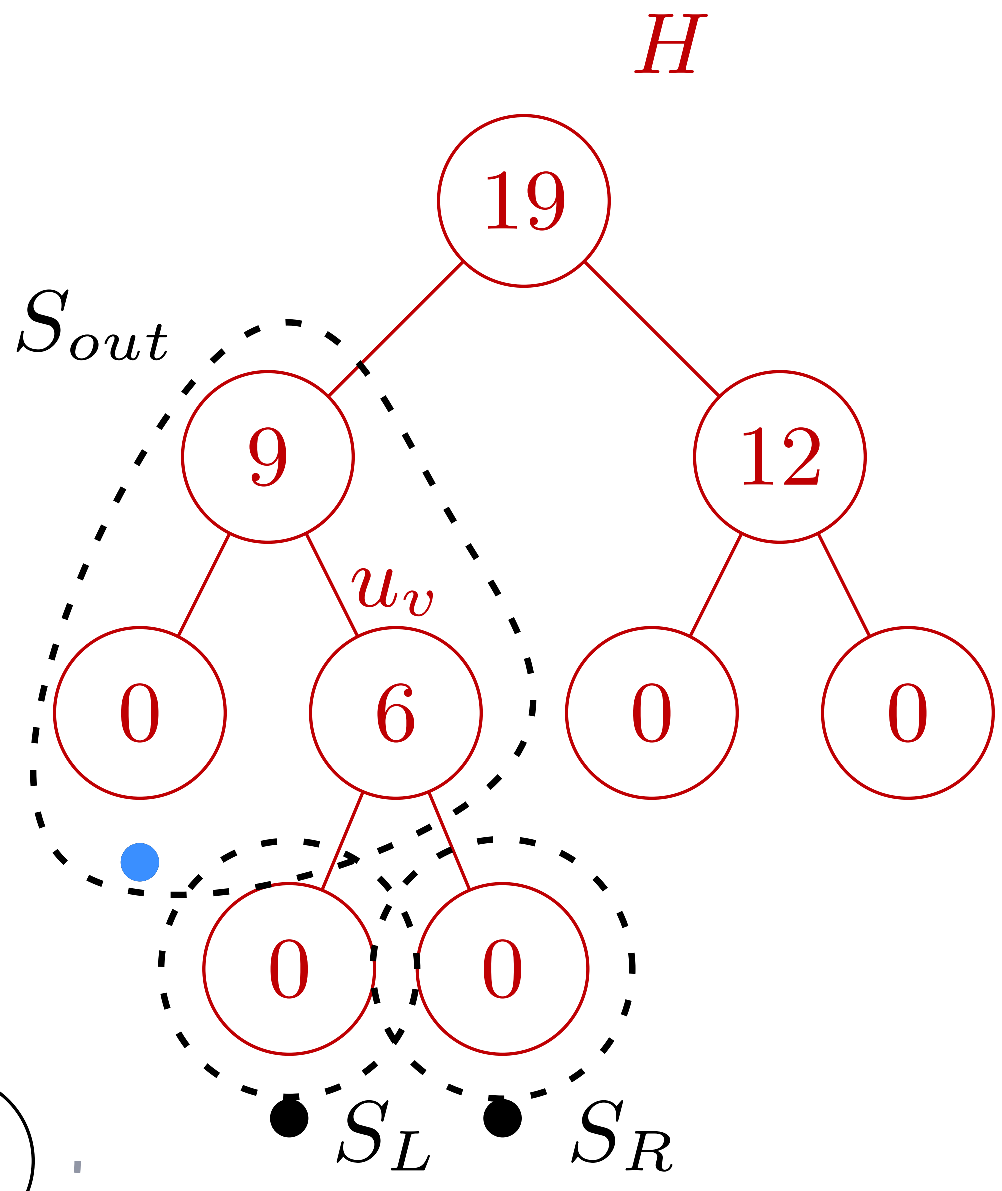
- $d(q, P^v) \in (r_v, R_v]$: Then the query finds a $(1+\varepsilon/4)$-ANN point $s \in P^v$ and returns it as the answer to the query
- $d(q, P^v) > R_v$:
  Then the search continues recursively in v_{out}





Correctness

Lemma: *The point returned by the data structure is a* $(1+\varepsilon)$*-ANN to the query point* q *in* P.

Proof (sketch, case 1):
$d(q, P^v) \leq r_v$; recurse into subtree with returned point p.

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We need to show that the algorithm recurses into a subtree which contains a $(1+\varepsilon)$-ANN point. If the algorithm continues to v_L, we have that $d(P_L^v, P^v \setminus P_L^v) \geq \Delta(u^v)/t$. For $q_L = nn(q, P_L^v)$, by the triangle inequality we have that

$$d(q, P_v \setminus P_L^v) \geq d(q_L, P^v \setminus P_L^v) - d(q, q_L) \geq \frac{\Delta(u^v)}{t} - r_v > \frac{\Delta(u^v)}{2t} > r_v$$

$$r_v = \Delta(u^v)/4t$$

Space Complexity

Lemma: *For $t = n^{O(1)}$, the data structure is made out of $O(\frac{n}{\varepsilon}\log^2 n)$ balls.*

Proof (sketch):

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Proof (sketch):

Let $U(n_v)$ be the number of balls used in $\hat{\mathcal{I}}_v$, we have $\mu = O(\varepsilon^{-1}\log n)$ and $\hat{\mathcal{I}}(P, a, b, \varepsilon)$ is made out of $O(\frac{n}{\varepsilon}\log(b/a))$ balls (previous slide).

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we have:

$$U(n_v) = O(\frac{n_v}{\varepsilon} \log \frac{R_v}{r_v}) = O(\frac{n_v}{\varepsilon} \log(\frac{t \log n}{\varepsilon}))$$

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we have:

$$U(n_v) = O(\frac{n_v}{\varepsilon}\log\frac{R_v}{r_v}) = O(\frac{n_v}{\varepsilon}\log(\frac{t\log n}{\varepsilon}))$$

count each occurrence of point as repr. (overall $2n$)

For the total number of balls, we get the recurrence
$B(n) = U(n) + B(n_L) + B(n_R) + B(n_{out})$
(we have $n_L, n_R, n_{out} \leq n/2 + 1$ and $B(n_L) + B(n_R) + B(n_{out}) = n$).
This results in $B(2n) = O(\varepsilon^{-1} n \log n \log(\varepsilon^{-1} t \log n)) = O((n/\varepsilon)\log^2 n)$

Number of Queries

Lemma: *For* $t = n^{O(1)}$*, the ANN-query algorithm performs* $O(\log(n/\varepsilon))$ *near neighbor queries*

Proof (sketch):

Number of Queries

Lemma: *For* $t = n^{O(1)}$*, the ANN-query algorithm performs* $O(\log(n/\varepsilon))$ *near neighbor queries*

Proof (sketch):

For every internal node v on the search path π in T corresponds to a situation where $d(q, P) \leq r_v$ or $d(q, P) > R_v$, which can be decided by two nearest neighbor queries.

In the final node u in π, the search algorithm resolves the query using $O\left(\log \frac{\log R_v / r_v}{\varepsilon}\right)$ $= O(\log(1/\varepsilon) + \log\log n)$ near neighbor queries.

Since the depth of the tree is $O(\log n)$, the total number of queries becomes $O(\log n + \log(1/\varepsilon) + \log\log n) = O(\log n/\varepsilon)$.

Summary

Approximating a metric space by a hierarchical well-separated tree (HST)

hierachical well-separated trees

simple $(n-1)$-approximation

fast $n^{O(1)}$-approximation in $\mathbb{R}^d$

ANN via point location among balls

simple construction

handling a range of radii

ANN data structure based on HST

. . . **and next time**

point location among approximate balls

approximate Voronoi diagrams